

FEASIBLE OBJECTIVE CONFIDENCE BOUNDS  
FOR SYSTEM RELIABILITY

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## ABSTRACT

The basic problem of determining objective (frequentistic) confidence bounds for the reliability of a series system based on failure data from tests of the independent components is addressed. The notion of confidence bounds based on orderings imposed on the sample space is exploited, and certain optimality considerations are incorporated. Advantage is taken of the simplifications resulting from the use of the Poisson approximation for data from highly reliable components. Tables of exact confidence bounds are produced for the case of two-component systems. These bounds are computed using sample orderings generated sequentially by a two-stage, prospective optimization procedure. A generalization of the Lindstrom-Madden technique is proposed for using the tables to find confidence bounds for systems consisting of more than two components with differing sample sizes.

Key Words: Reliability, series-system, confidence bounds, Poisson approximation, Lindstrom-Madden, sample orderings.

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## 1. Introduction

Certainly one of the most basic statistical problems in the assessment of system reliability is that of determining a confidence bound on the reliability of a series system based on component failure data. The continuing appearance of papers concerned with this subject (e.g., Harris and Soms 1981, Butcher et al. 1978, and Winterbottom 1980) testifies to the elusiveness of solutions which are both practically feasible and acceptably precise. The present paper deals with the case of systems characterized by high intrinsic reliability ( $>90\%$ ) where the use of the Poisson approximation for the binomial distributions of component failure data does not introduce appreciable error. The emphasis is on objective, frequentistic confidence bounds which avoid the uncertainties of interpretation associated with posterior bounds obtained by Bayesian methods. The use of the Poisson distribution provides several advantages. It easily leads to valid results for cases involving zero observed failures for some components where maximum likelihood methods and other proposed approximations tend to break down (see, e.g., Madansky 1965 and Mann et al. 1974). It also permits the pooling of failure data for different components having the same test sample sizes. This potential reduction in the effective number of system components enhances the utility of tabulated bounds such as those presented here.

Because of the structure of the problem, universally optimal confidence bounds for system reliability (i.e., "uniformly most accurate" bounds in the sense of Lehmann 1959) do not generally exist. On the other hand, the ideas of Buehler (1957) may be exploited to produce a variety of valid confidence bounds based on total orderings of the sample points. Such bounds are exact in the sense that the desired coverage probability is guaranteed. The construction of good confidence bounds is thus reduced to the selection of suitable orderings imposed on the sample space. This is the approach adopted in the present study. Previous applications of these methods to reliability may be found, for example, in Harris and Soms (1980), Johnson (1969), and Lipow and Riley (1959).

Once the sample ordering approach has been chosen, there remains the problem of determining orderings which lead to confidence bounds which are "good" according to some criterion measuring the size (length) of the confidence region. An early proposal of the present author (Johns 1975) was to generate the ordering by means of a simple function of the observations which was asymptotically equivalent to the maximum likelihood confidence bound. This method guarantees asymptotic optimality when the numbers of component failures observed under testing is large. It was found, however, that the resulting bounds could be noticeably improved for small to moderate numbers of observed failures by more sophisticated methods. Another procedure investigated (Johns 1977) was a sequential method for generating the sample ordering starting at the origin (zero component failures observed) and selecting at each stage as the next sample

point the "adjacent" point producing the largest value for the lower bound on reliability. This method, while intuitively appealing, does not generally produce a "best" ordering as has been suggested by some investigators. In particular, it is improved upon by the method adopted in the present paper.

The non-existence of a unique optimal ordering leaves open the possibility of obtaining a confidence bound which is at least admissible by choosing the sample ordering to minimize the expected length of the confidence interval under some suitable prior distribution. Such a semi-Bayesian approach does not in any way impair the frequentistic interpretation of the confidence bounds obtained from the minimizing ordering. The implementation of such a minimization, while theoretically perfectly possible, turns out to be totally unfeasible computationally except for the earliest part of the ordering generated. Nevertheless, for a class of priors chosen to emulate certain properties of maximum likelihood, fragmentary orderings computed by this method provide a considerable justification for the two-stage "look-ahead" sequential method actually used to generate the tables which are a principal concern of this paper.

Suppose that the series system under consideration consists of  $k$  independent components and that the respective probabilities of component failure are  $p_i = 1 - q_i$ ,  $i = 1, 2, \dots, k$ . The system reliability  $R$  is given by

$$R = \prod_{i=1}^k q_i = \prod_{i=1}^k (1 - p_i) \quad . \quad (1.1)$$



If the observed numbers of failures for the  $k$  components are  $X_1, X_2, \dots, X_k$  based on independent tests with corresponding sample sizes  $n_1, n_2, \dots, n_k$ , we let  $\lambda_i = n_i p_i$ ,  $i = 1, 2, \dots, k$ , so that

$$R = \prod_{i=1}^k (1 - \lambda_i/n_i) \approx 1 - \sum_{i=1}^k \lambda_i/n_i . \quad (1.2)$$

The approximation on the right will be best when the  $p_i$ 's ( $= \lambda_i/n_i$ ) are all small which is just the case where  $R$  is close to one and the Poisson approximation for the distributions of the  $X_i$ 's is valid.

It will be convenient to express the problem in a canonical form by introducing some further notation. Let  $c = \sum_{i=1}^k 1/n_i$  and  $a_i = 1/cn_i$ ,  $i = 1, 2, \dots, k$ . Then letting  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  and

$$\theta(\lambda) = \sum_{i=1}^k a_i \lambda_i , \quad (1.3)$$

we have

$$R \approx 1 - c \sum_{i=1}^k a_i \lambda_i = 1 - c\theta(\lambda) . \quad (1.4)$$

We shall assume henceforth that the components are indexed so that  $n_1 \geq n_2 \geq \dots \geq n_k$  which implies that  $a_1 \leq a_2 \leq \dots \leq a_k$ . The problem of finding a lower confidence bound for  $R$  is thus reduced to that of finding an upper confidence bound for  $\theta(\lambda)$ . The fact that  $\theta(\lambda)$  is a convex combination of the  $\lambda_i$ 's facilitates tabulation of the bounds by reducing the number of classification variables by one. In principle, confidence bounds for  $R$  could be constructed directly without introducing the approximation (1.2). Such an approach would,

however, eliminate the possibility of constructing useful tables of bounds, since separate entries would be required for every configuration of  $(n_1, n_2, \dots, n_k)$ .

The vector of observations is  $X = (X_1, X_2, \dots, X_k)$  so that the sample space on which a total ordering must be imposed consists of all vectors  $x = (x_1, x_2, \dots, x_k)$  where the  $x_i$ 's are non-negative integers. It seems reasonable to confine attention to orderings which are consistent with the natural partial ordering induced by dominance (see Section 2), and we shall do so. In Section 2 it is shown that the best upper confidence bound for  $\theta(\lambda)$  with confidence coefficient  $1 - \alpha$  which is monotone in a prescribed total ordering (designated by the relation  $\leq$ ) is given by

$$t(x) = \sup_{\lambda \in S_{\alpha}^*(x)} \theta(\lambda) \quad , \quad (1.5)$$

where  $S_{\alpha}^*(x) = \{\lambda : P_{\lambda}\{X \leq x\} = \alpha\}$ . From (1.4) the lower confidence bound for system reliability is then given by

$$r(x) = 1 - ct(x) \quad . \quad (1.6)$$



Because of the additive property of independent Poisson observations the effective number of components in the system may be reduced if some sample sizes are equal (or nearly equal) as follows: Suppose  $n_1 = n_2 = \dots = n_r = n$  for some  $r$ ,  $2 \leq r < k$ . Then  $a_1 = a_2 = \dots = a_r = a^*$  (say) and letting  $\lambda^* = \sum_{i=1}^r \lambda_i$  we have

$$\theta(\lambda) = a^* \lambda^* + \sum_{i=r+1}^k a_i \lambda_i. \quad (1.7)$$

Now  $X^* = \sum_{i=1}^r X_i$  has a Poisson distribution with parameter  $\lambda^*$  so that the right hand side of (1.7) has the correct form for the case of dimension  $k^* = k - r + 1$  except that the coefficients must each be divided by  $c_0 = a^* + a_{r+1} + \dots + a_k$  to preserve convexity. The confidence bound for the  $k^*$  dimensional case based on  $X^*, X_{r+1}, \dots, X_k$  and the normalized coefficients may then be multiplied by  $c_0$  to obtain the bound for the original  $\theta(\lambda)$  given by (1.7). Further reductions may be made in the same way if several groups of components have common sample sizes. If  $n_1, n_2, \dots, n_r$  are only approximately equal, their average may be used for  $n$  in the above calculations to obtain an approximate bound.

If all components are subjected to the same number of trials, we may take  $k$  equal to one and the problem reduces to the familiar one of finding an upper confidence bound for a single Poisson parameter.

The component failure data may be developed through independent testing of the components, or through testing of the complete system with the assignment of failures to the appropriate components.

Even in the latter case component sample sizes may differ if components are redesigned during the course of testing so that the trials and failures observed prior to redesign are not relevant to the reliability of the final version of the system.

It is worth noting that the same formal confidence bound problem arises from a somewhat different and more restrictive model involving continuous time sampling and exponentially distributed failure times. Thus, if we assume that the time to failure of the  $i$ -th component has the exponential probability density function  $\mu_i e^{-\mu_i t}$ ,  $\mu_i > 0$ ,  $t > 0$ , and if the fixed aggregate testing time for the  $i$ -th component is given by  $\tau_i$  and the total number of failures observed is  $X_i$ ,  $i=1,2,\dots,k$ , then the  $X_i$ 's which are sufficient statistics have independent Poisson distributions with corresponding parameters  $\lambda_i = \mu_i \tau_i$ . Such test data would be generated, for example, if a single unit of each component were placed on test and repaired whenever a failure occurred until the total test time for each component reached its corresponding limit,  $\tau_i$ . The system reliability for a single duty cycle (consisting of operation for a unit interval of time) is given by

$$R = \prod_{i=1}^k e^{-\mu_i} = \exp \left\{ - \sum_{i=1}^k \mu_i \right\} = \exp \left\{ - \sum_{i=1}^k \lambda_i / \tau_i \right\}, \quad (1.8)$$

and the confidence bound problem is essentially the same as the one previously introduced except that no approximations are required.

The general theory of confidence bounds based on sample orderings is discussed in Section 2. In Section 3 the case of systems having two components is considered in detail, and the rationale for, and use of, the tables for this case are explained. Section 4 includes suggestions for constructing approximate confidence bounds for cases of systems with  $k \geq 3$  by finding approximately equivalent cases with  $k = 2$ . Use of the maximum likelihood ratio bounds for cases where the data are beyond the limits of the available tables is also discussed.

## 2. Bounds and Orderings: Generalities

The idea of using sample orderings to generate confidence bounds was first introduced by Buehler (1957) who discussed the validity of the proposed method in the context of a specific reliability problem. Bol'shev and Loginov (1969) discuss the construction of confidence bounds monotone in the sample orderings generated by certain functions of the observations. In the following, which is a revision and extension of Johns (1975), we develop the theory with emphasis on the sample orderings themselves rather than possible generators of the orderings.

To develop the general ideas relating exact confidence bounds to sample orderings it is convenient to introduce a fairly abstract statistical model. Let the sample space  $\mathcal{X}$  be endowed with a measurable total ordering relation " $\leq$ " and let  $X$  represent the random outcome of the experiment where the space of possible outcomes is  $\mathcal{X}$ . Suppose that the possible distributions of  $X$  are determined by the family of probability measures  $P_\lambda$ , indexed by  $\lambda$ , an element of the parameter space  $\Lambda$ . Our objective is to find a  $1 - \alpha$  level upper confidence bound for a specified real-valued function  $\theta(\lambda)$  defined on  $\Lambda$  where the range  $T$  of  $\theta(\lambda)$  is assumed to be closed and bounded below. The quantity  $\alpha \in (0,1)$  is regarded as fixed throughout. We make the following definitions and assumptions:

Definition D1. For each  $x \in \mathcal{X}$  let

$$S_\alpha(x) = \{\lambda : P_\lambda\{X \leq x\} > \alpha\} . \quad (2.1)$$

Definition D2. For each  $x \in \mathcal{X}$  let

$$t(x) = \begin{cases} \sup\{\theta(\lambda) : \lambda \in S_\alpha(x)\} , & \text{if } S_\alpha(x) \text{ is non-empty} , \\ \inf T, & \text{otherwise} . \end{cases} \quad (2.2)$$

Remark 1. By D2 if  $\theta(\lambda) > t(x)$ , then necessarily  $P_\lambda\{X \leq x\} \leq \alpha$ .

Assumption A1. For every subset  $C$  of  $\mathcal{X}$  having the property that if  $x \in C$  and  $y \leq x$  then  $y \in C$ , there exists an ordered sequence  $x_1 \leq x_2 \leq \dots$  of elements of  $C$  such that  $C = \bigcup_{n=1}^{\infty} \{x : x \leq x_n\}$ .

Assumption A2. For each  $x \in \mathcal{X}$ , if  $\theta(\lambda) = t(x)$ , then  $P_\lambda\{X \leq x\} \leq \alpha$ .

Remark 2. By D1, D2, and A2 if  $\theta(\lambda) = t(x)$ , then  $\lambda \notin S_\alpha(x)$ , i.e., the supremum in D2 is never attained.

We now establish the following propositions.

Proposition P1. The function  $t(x)$  is monotone in the ordering on  $\mathcal{X}$ .

Proof: If  $x, y \in \mathcal{X}$  and  $x \leq y$ , then  $S_\alpha(x) \subset S_\alpha(y)$  (D1) which in turn implies  $t(x) \leq t(y)$  (D2).  $\square$

Proposition P2. Under assumptions A1 and A2 the function  $t(x)$  is an upper confidence bound for  $\theta(\lambda)$  at level  $1 - \alpha$ . In particular,

$$P_\lambda\{\theta(\lambda) < t(X)\} \leq \alpha \quad \text{for all } \lambda \in \Lambda. \quad (2.3)$$

Proof: For arbitrary  $\lambda_0 \in \Lambda$ , let  $\theta_0 = \theta(\lambda_0)$  and  $C_0 = \{x: t(x) \leq \theta_0\}$ .

The result follows immediately for all  $\lambda_0$  for which  $C_0$  is empty.

Assume that  $C_0$  is non-empty. Then by P1 the set  $C_0$  possesses the property required in A1 for the existence of a sequence  $\{x_n\} \subset C_0$  such that  $x_n \leq x_{n+1}$  for all  $n$ , and  $C_0 = \bigcup_{n=1}^{\infty} \{x: x \leq x_n\}$ . This implies that, as  $n \rightarrow \infty$ ,

$$P_{\lambda_0}\{X \leq x_n\} \uparrow P_{\lambda_0}\{X \in C_0\}. \quad (2.4)$$

But by Remark 1 and A2, for all  $n$ ,  $P_{\lambda_0}\{X \leq x_n\} < \alpha$ . Hence

$P_{\lambda_0}\{X \in C_0\} \leq \alpha$  and the desired result follows.  $\square$

Proposition P3. Under assumptions A1 and A2, if  $\tilde{t}(x)$  is any T-valued confidence bound such that  $P_\lambda\{\theta(\lambda) < \tilde{t}(X)\} \geq 1 - \alpha$  for all  $\lambda \in \Lambda$ , then

$$(i) \quad \sup_{y \leq x} \tilde{t}(y) \geq t(x) \quad \text{for all } x \in \mathcal{X}, \quad \text{and}$$

(ii) if  $t(x)$  is monotone in the ordering on  $\mathcal{X}$ , then

$$\tilde{t}(x) \geq t(x) \quad \text{for all } x \in \mathcal{X}.$$

Proof: First we assume that  $\tilde{t}(x)$  is monotone and establish (ii).

Suppose there exists an  $x' \in \mathcal{X}$  such that  $\tilde{t}(x') < t(x')$ . Then  $S_\alpha(x')$  must be non-empty and by Remark 2 following A2, the sup defining  $t(x')$  is not attained. Hence there exists a  $\lambda' \in S_\alpha(x')$  such that  $\tilde{t}(x') < \theta(\lambda') < t(x')$ , and  $P_{\lambda'}\{X \leq x'\} > \alpha$ . Thus by the monotonicity of  $\tilde{t}(x)$ ,

$$P_{\lambda'}\{\tilde{t}(X) \leq \theta(\lambda')\} \geq P_{\lambda'}\{\tilde{t}(X) \leq \tilde{t}(x')\} = P_{\lambda'}\{X \leq x'\} > \alpha. \quad (2.5)$$

This contradicts the hypothesis that  $P_\lambda\{\theta(\lambda) < \tilde{t}(X)\} \geq 1 - \alpha$  for all  $\lambda$  and establishes (ii). To show (i) we let  $t^*(x) = \sup_{y \leq x} \tilde{t}(y)$  and observe that  $t^*(x)$  is monotone in the ordering on  $\mathcal{X}$  and  $t^*(x) \geq \tilde{t}(x)$  for all  $x \in \mathcal{X}$ . Hence if  $\tilde{t}(x)$  is a  $1 - \alpha$  confidence bound for  $\theta(\lambda)$ , so is  $t^*(x)$  and applying (ii) to  $t^*(x)$  yields (i).  $\square$ .

In order to specialize these results in the direction of applications we henceforth assume that the parameter  $\lambda$  and the

observation  $X$  are both of dimension  $k$ , i.e.,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  and  $X = (X_1, X_2, \dots, X_k)$  where the  $\lambda_i$ 's are real and the  $X_i$ 's are random variables. Without essential loss of generality we assume that  $\Lambda$  contains the positive orthant. Within this framework we make the following additional assumptions:

Assumption A3. The function  $\theta(\lambda)$  is continuous and strictly increasing in each of the  $\lambda_i$ 's.

Assumption A4. For any  $x \in \mathcal{X}$ ,  $P_\lambda\{X \leq x\}$  is continuous in each of the  $\lambda_i$ 's.

Proposition P4. Assumptions A3 and A4 imply that Assumption A2 is satisfied.

Proof: Suppose that for some  $x' \in \mathcal{X}$  there exists a  $\lambda' \in \Lambda$  such that  $\theta(\lambda') = t(x')$  and  $P_{\lambda'}\{X \leq x'\} > \alpha$ . Then by A3 and A4 we can find a  $\lambda'' \in \Lambda$  with  $\lambda_i'' \geq \lambda_i'$  for all  $i$  and  $\lambda_{i_0}'' > \lambda_{i_0}'$  for some  $i_0$  such that  $\theta(\lambda'') > t(x')$  and  $P_{\lambda''}\{X \leq x'\} > \alpha$ . This contradicts D2 and the result follows.  $\square$

Corollary C1. If (i) the observations  $X_1, X_2, \dots, X_k$  are independent Poisson random variables with parameters  $\lambda_1, \lambda_2, \dots, \lambda_k$  respectively, and (ii)  $\theta(\lambda) = a_1\lambda_1 + a_2\lambda_2 + \dots + a_k\lambda_k$  where the  $a_i$ 's are positive, then  $t(x)$  given by D2 is an upper confidence bound for  $\theta(\lambda)$  at level  $1-\alpha$ .



Proof: A1 is satisfied for any ordering since  $\mathcal{X}$  is discrete. A3 is clearly satisfied for  $\theta(\lambda)$  of the form given by (ii), and A4 is satisfied for Poisson random variables. The desired result follows from P4 and P2.  $\square$

The actual computation of the bound  $t(x)$  given by D2 is greatly facilitated if  $P_\lambda\{X \leq x\}$  is monotone in the components of  $\lambda$ . The following proposition gives conditions guaranteeing this property:

Assumption A5. The components of  $X$  are independent and for  $i=1,2,\dots,k$  the distribution of each  $X_i$  depends only on the corresponding  $\lambda_i$  and is stochastically increasing in  $\lambda_i$ .

Definition D3. We denote by  $(\preceq^*)$  the natural partial ordering of  $\mathcal{X}$  generated by componentwise dominance. That is,  $x \preceq^* y$  if and only if  $x_i \leq y_i$  for  $i=1,2,\dots,k$ , with  $x \prec^* y$  if at least one of these inequalities is strict. An arbitrary total ordering  $(\preceq)$  on  $\mathcal{X}$  is consistent with  $(\preceq^*)$  if  $x \preceq^* y$  implies  $x \preceq y$ .

Proposition P5. If the ordering on  $\mathcal{X}$  is consistent with the natural partial ordering and Assumption A5 is satisfied, then for each  $x \in \mathcal{X}$ ,  $P_\lambda\{X \leq x\}$  is non-increasing in each component of  $\lambda$ .

Proof: For  $y \in \mathcal{X}$  we introduce the representation  $y = (y_1, y^{(2)})$  where  $y^{(2)} = (y_2, y_3, \dots, y_k)$ . For real  $z$ , fixed  $x \in \mathcal{X}$ , and all  $y \in \mathcal{X}$ , let  $I_x(z, y^{(2)})$  be the indicator function of the set  $\{y^{(2)} : (z, y^{(2)}) \leq x\}$ .

Then

$$P_{\lambda}\{X \leq x\} = E_{\lambda} I_x(X_1, X^{(2)}) \quad (2.6)$$

For any  $y \in \mathcal{Y}$ , if  $z' \leq z''$ , then  $(z', y^{(2)}) \leq^* (z'', y^{(2)})$ , by the consistency hypothesis, and  $I_x(z', y^{(2)}) \geq I_x(z'', y^{(2)})$ . Hence letting  $G_x(z) = E_{\lambda}\{I_x(X_1, X^{(2)}) | X_1 = z\}$  we see that  $G_x(z)$  is non-increasing in  $z$ . Thus, since the distribution of  $X_1$  is stochastically increasing in  $\lambda_1$  (A5), we conclude that  $E_{\lambda} I_x(X_1, X^{(2)}) = E_{\lambda} G_x(X_1)$  is non-increasing in  $\lambda_1$ . The same argument applies to the other components of  $\lambda$  establishing the desired result.  $\square$

Suppose that  $\Lambda$  is the non-negative orthant of  $R^{(k)}$  and let  $\mathcal{S}$  be the simplex  $\mathcal{S} = \{\lambda : \sum_{i=1}^k \lambda_i = 1\}$ . If A4 and A5 are satisfied, then by P5 we observe that for any  $\lambda \in \mathcal{S}$  and real  $c$ ,  $P_{c\lambda}\{X \leq x\}$  is continuous and non-increasing in  $c$ . If the lower bound of  $P_{c\lambda}\{X \leq x\}$  as  $c \rightarrow \infty$  is less than  $\alpha$  for all  $x \in \mathcal{X}$  and all  $\lambda \in \mathcal{S}$ , then for each  $x \in \mathcal{X}$  and  $\lambda \in \mathcal{S}$  there exists a smallest number  $b = b(x, \lambda)$  such that  $P_{b(x, \lambda)\lambda}\{X \leq x\} = \alpha$ . The confidence bound  $t(x)$  defined by D2 is then given by

$$t(x) = \sup_{\lambda \in \mathcal{S}} \theta(b(x, \lambda)\lambda) \quad (2.7)$$

Now  $b(x, \lambda)$  is easily computed using root-finding techniques so that the computation of  $t(x)$  reduces to searching over  $\mathcal{S}$  for the maximum of  $\theta(b(x, \lambda)\lambda)$ . Many routines are available for implementing such searches. For the situation described in Corollary C1, the value of

b such that  $P_{b\lambda}\{X \leq x\} = \alpha$  is unique and (2.7) is a computationally feasible version of (1.5).

All of the above results apply mutatis mutandis to the construction of lower confidence bounds and hence confidence intervals. Applications to the reliability of coherent systems involving the binomial or other distributions are possible. In particular, the above discussion applies directly to the binomial case under the transformation  $\lambda_i = -\log(1-p_i)$ ,  $i = 1, 2, \dots, k$ ;  
 $\theta(\lambda) = \sum_{i=1}^k \lambda_i = -\log \prod_{i=1}^k (1-p_i)$ .

### 3. Systems With $k = 2$

As was noted in Section 1, if the system has effectively only one component (e.g., when all sample sizes are equal), then the problem reduces to the well-known case of finding an upper confidence bound for a single Poisson parameter. Then in the notation of Section 1,  $\theta(\lambda) = \lambda$  and if  $t(x)$  is the confidence bound for  $\lambda$ , the lower confidence bound (1.6) for reliability  $R$  becomes

$$r(x) = 1 - t(x)/n, \quad (3.1)$$

where  $n$  is the (common) sample size.

The two component case ( $k = 2$ ) presents all of the difficulties of the general case. The principal problem is to generate an ordering of the sample points  $x = (x_1, x_2)$  which will lead to a "good" confidence bound  $t(x)$  computed using (1.5) or (2.7). Several different methods have been considered and implemented to varying extents

in the course of this investigation. These methods may be described briefly as follows:

- (i) The  $x$ 's are ordered according to the values of the function  $\tilde{t}(x) = a_1x_1 + a_2x_2 + z_\alpha \sqrt{a_1^2x_1^2 + a_2^2x_2^2}$ , where  $z_\alpha$  is the upper  $\alpha$ -th quantile of the standard normal distribution.
- (ii) The  $x$ 's are ordered according to the values of the approximate confidence bound obtained from the maximum likelihood ratio statistic (see Section 4).
- (iii) The ordering is generated sequentially by considering at each stage the group of points which are not yet ordered but could be adjoined without violating the natural partial ordering (see D3 of Section 2). The next point in the ordering is then chosen to be the "best" member of the candidate group, i.e., the point producing the smallest value of  $t(x)$  given by (1.5).
- (iv) The ordering is chosen so as to minimize  $E_G\{t(X)\}$  for some suitable prior distribution  $G$  over the values of  $\lambda$ .
- (v) The ordering is generated sequentially in the manner of (iii) above except that at each stage the candidate points for the next two steps are considered as pairs and the next point selected is the first step component which, together with the best available point for the second step, produces the smallest sum for the two values of  $t(x)$ . Note that the point that appears to be "best" two steps ahead may not actually be chosen when that stage is reached.

It is clear that none of these methods is special to the case  $k = 2$ . Method (i), based on the function  $\tilde{t}(x)$ , which is really a maximum likelihood estimate of an asymptotically valid confidence bound, was used to generate tables of bounds for the case  $k = 2$  in Johns (1975). Method (ii) does not improve substantially on Method (i) for moderate values of the  $X_i$ 's. Methods (i) and (ii) are asymptotically equivalent when at least one  $X_i$  becomes large (see Johns 1975) and indeed standard maximum likelihood results guarantee that both are asymptotically optimal. Method (iii) discussed in Johns (1977) was found to be a substantial improvement on (i) in the strong sense that when the Method (iii) ordering is used the values of  $t(x)$  are often smaller and only rarely slightly larger than the values for corresponding  $x$ 's produced by Method (i).

The semi-Bayesian approach of Method (iv), which minimizes the expected length of the confidence interval, is the only one of the five that is directly motivated by optimality considerations. The bound resulting from any reasonable prior must at least be admissible. In pursuing this approach it was decided in the spirit of objectivity and in the hope of rapid convergence to asymptotic optimality to choose a prior distribution leading to an unconditional probability mass function for  $x_1$  and  $x_2$  constant for constant values of the maximum likelihood estimator  $a_1x_1 + a_2x_2$  for  $\theta(\lambda)$ . In particular, the prior density for  $\lambda_1$  and  $\lambda_2$  was taken to be

$$g(\lambda_1, \lambda_2) = b_1 b_2 e^{-b_1 \lambda_1 - b_2 \lambda_2}, \lambda_1, \lambda_2 > 0, \quad (3.2)$$

where  $b_1 = (1 - e^{-\beta a_1})$ ,  $b_2 = (1 - e^{-\beta a_2})$ ,  $\beta > 0$ . This produces the unconditional probability mass function

$$p(x_1, x_2) = b_1 b_2 e^{-\beta(a_1 x_1 + a_2 x_2)}, \quad x_1, x_2 = 0, 1, \dots, \quad (3.3)$$

In the limiting case, as  $\beta \rightarrow 0$ ,  $p(x_1, x_2)$  becomes essentially uniform over any finite set of points  $(x_1, x_2)$ .

The actual minimization of  $E_G\{t(X)\}$  may, in principle, be accomplished by finding the ordering which minimizes the contribution to  $E_G$  among all orderings of length  $N$  where  $N$  may be arbitrarily large. This may be done systematically by starting at the origin  $(0, 0)$  and constructing a tree whose nodes at each stage are characterized by a candidate point newly adjoined to the ordering and the corresponding value of  $\sum p(x)t(x)$ , where the sum is taken over all  $x$ 's occurring in the path leading to the node, including the one just adjoined. At the  $N$ -th stage the node having the smallest accumulated sum identifies the optimal ordering of length  $N$ . This process may be facilitated by eliminating duplicate nodes and discontinuing branches when a node is reached whose value exceeds that known to be attainable in  $N$  stages. Nevertheless, because of the rapid increase in the number of nodes considered per stage, only the first forty or so points in the optimal orderings could be determined even using a very large computer facility.

In order to obtain examples of admissible orderings with which to compare the results of other methods, this computation was performed for two cases using an IBM 370/168. For both cases the

values  $\alpha = .10$  and  $a_1 = .30$  were used. For the first case the probabilities given in (3.3) with  $\beta = 1$  were used and the first 41 points of the optimal ordering were obtained. The forty-first stage of the computation produced 4557 nodes. For the second case the limiting situation as  $\beta \rightarrow 0$  where the  $p(x)$ 's are all equal was used and the first 43 points of the optimal ordering were obtained. The number of nodes produced at the forty-third stage was 6478.

A comparison of these results with the corresponding results obtained using Methods (i), (iii), and (v) is indicated in Figure 1. The horizontal axis indexes the first 50 points in the ordering produced by Method (v), the two-step prospective sequential procedure. The values of  $t(x)$  for these indexed points for the five methods are indicated by the plotted symbols. Values of  $t(x)$  for methods other than (v) are shown only when they differ from those produced by that method. Based on this evidence it appears that Method (iv) and Method (v) differ very little and that both are better than the other methods. In fact, Method (iv) for the uniform case ( $\beta = 0$ ) differs only trivially from Method (v). Since the use of Method (iv) for the construction of tables is now and probably always will be impractical, we are led to the choice for this purpose of the more tractable and virtually equivalent Method (v). Prospective sequential methods looking ahead more than two steps might be feasible, although the complexity of the computations increases rapidly with the number of steps. However, such procedures would be expected to produce only minute improvement over the two-step method.



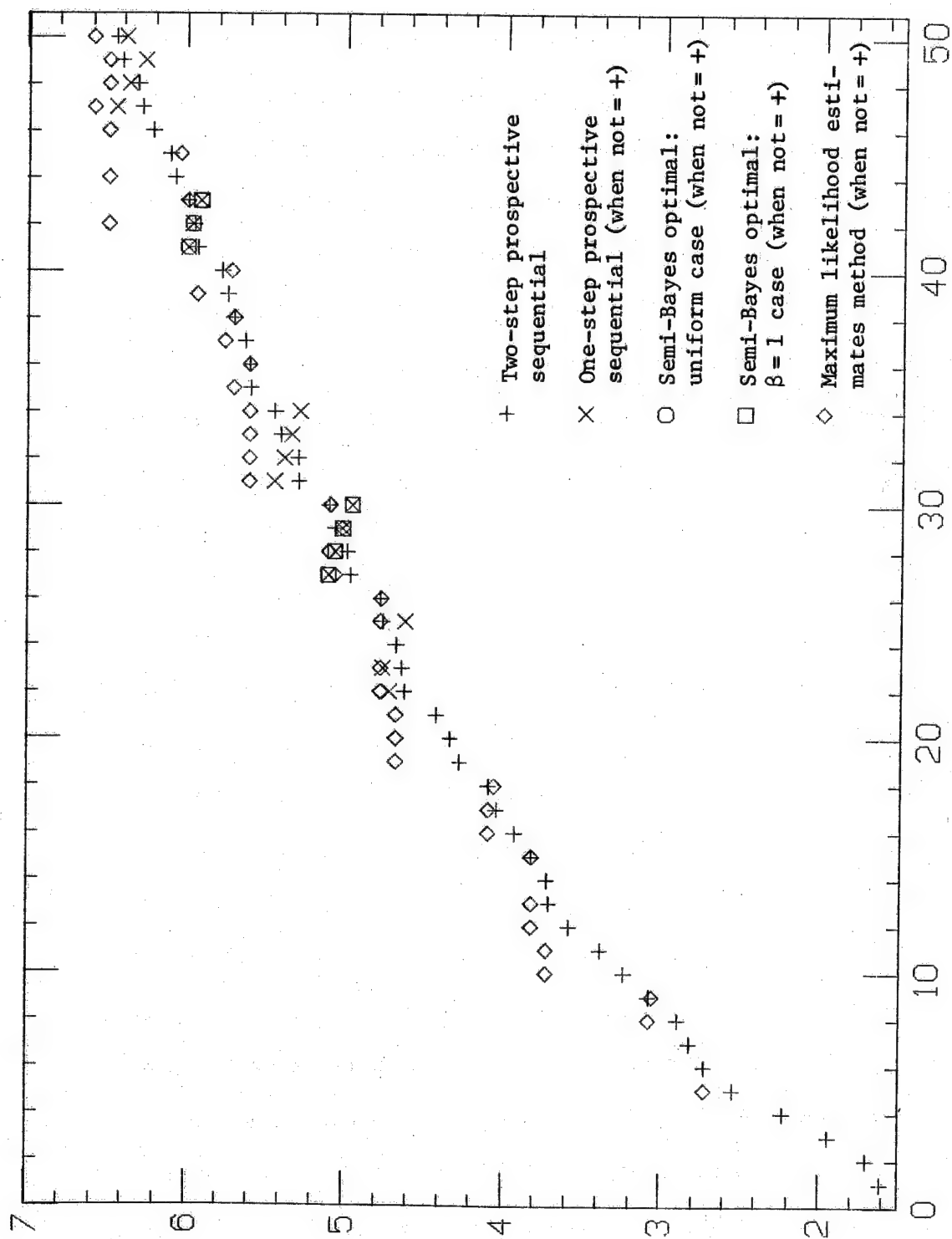


Figure 1. Comparison of confidence bounds for five ordering methods:  $\alpha = .10$ ,  $a_1 = .30$

Table 1 gives values of the upper confidence bound  $t(x)$  computed using (2.7) with the sample orderings generated by the two-step prospective sequential Method (v). The values are given for the first 100 points in each ordering for  $\alpha = .01, .05, .10$ , for  $a_1 = .05(.05).45$ . It is assumed that the components of the system are indexed so that  $a_1 < a_2$  which implies  $a_1 < .50$ . For convenience the values of  $(x_1, x_2)$  are listed systematically rather than in the order generated by the two-step procedure. This table provides a basis for computing accurate confidence bounds for the case  $k = 2$  using only simple interpolation. If values of  $a_1$  greater than .45 but (necessarily) less than .50 are required, the bound for  $a_1 = .5$  (corresponding to  $n_1 = n_2$ ) may be used for interpolation. This bound is obtained by simply multiplying the ordinary upper confidence bound for a single Poisson parameter based on  $x = x_1 + x_2$  failures by .50 (see, e.g., Pearson and Hartley 1958 for tables). The use of Table 1 is illustrated by the following two examples:

Example 1. Suppose that the two components of a series system are tested independently using sample sizes  $n_1 = 300$  and  $n_2 = 100$  respectively with the corresponding observed numbers of failures  $X_1 = 3$  and  $X_2 = 4$ . Then  $c = (1/300 + 1/100) = 4/300$  and  $a_1 = 1/cn_1 = .25 = 1 - a_2$ . If we wish to find a 95 percent confidence interval, we take  $\alpha = .05$ , and from Table 1 we find the confidence bound  $t(x)$  for  $\theta(\lambda)$  to be 7.333. Hence by (1.6) the 95 percent lower confidence bound for system reliability  $R$  is  $1 - (4/300)(7.333) = .902$ .

Table 1. The Confidence Bound  $t(x)$  for  $k = 2$  for the First 100 Points Generated by the Two-Stage Optimal Ordering Method for Each  $a_1$  and  $\alpha$ .

$a_1 = 0.05$				$a_1 = 0.05$				$a_1 = 0.05$				$a_1 = 0.05$				$a_1 = 0.10$			
$x_1$	$x_2$	$t(x_1, x_2)$	$t(x_1, x_2)$	$x_1$	$x_2$	$t(x_1, x_2)$	$t(x_1, x_2)$	$x_1$	$x_2$	$t(x_1, x_2)$	$t(x_1, x_2)$	$x_1$	$x_2$	$t(x_1, x_2)$	$t(x_1, x_2)$	$x_1$	$x_2$	$t(x_1, x_2)$	$t(x_1, x_2)$
0	0	4.375	5.262	0	0	2.846	3.733	0	0	2.188	3.075	25	0	2.188	3.075	25	0	2.188	3.075
0	1	6.306	7.195	0	1	4.507	5.396	0	1	3.695	4.585	25	1	3.695	4.585	25	1	3.695	4.585
1	0	4.376	5.306	1	0	2.847	3.777	1	0	2.189	3.119	26	0	2.189	3.119	26	0	2.189	3.119
1	1	6.308	7.239	1	1	4.508	5.440	1	1	3.697	4.629	26	1	3.697	4.629	26	1	3.697	4.629
2	0	4.387	5.351	2	0	2.858	3.822	2	0	2.200	3.163	27	0	2.200	3.163	27	0	2.200	3.163
2	1	6.319	7.284	2	1	4.519	5.485	2	1	3.708	4.674	27	1	3.708	4.674	27	1	3.708	4.674
3	0	4.407	5.395	3	0	2.878	3.866	3	0	2.220	3.208	28	0	2.220	3.208	28	0	2.220	3.208
3	1	6.327	7.329	3	1	4.539	5.529	3	1	3.727	4.718	28	1	3.727	4.718	28	1	3.727	4.718
4	0	4.432	5.440	4	0	2.903	3.911	4	0	2.245	3.253	29	0	2.245	3.253	29	0	2.245	3.253
4	1	6.364	7.385	4	1	4.564	5.574	4	1	3.753	4.763	29	1	3.753	4.763	29	1	3.753	4.763
5	0	4.461	5.530	5	0	2.932	3.956	5	0	2.274	3.298	30	0	2.274	3.298	30	0	2.274	3.298
5	1	6.393	7.457	5	1	4.593	5.619	5	1	3.781	4.809	30	1	3.781	4.809	30	1	3.781	4.809
6	0	4.493	5.621	6	0	2.964	3.981	6	0	2.305	3.343	31	0	2.305	3.343	31	0	2.305	3.343
6	1	6.424	7.500	6	1	4.624	5.665	6	1	3.813	4.854	31	1	3.813	4.854	31	1	3.813	4.854
7	0	4.526	5.712	7	0	2.997	4.047	7	0	2.339	3.388	32	0	2.339	3.388	32	0	2.339	3.388
7	1	6.458	7.578	7	1	4.658	5.712	7	1	3.847	4.899	32	1	3.847	4.899	32	1	3.847	4.899
8	0	4.561	5.804	8	0	3.032	4.138	8	0	2.374	3.434	33	0	2.374	3.434	33	0	2.374	3.434
8	1	6.493	7.650	8	1	4.693	5.804	8	1	3.882	4.945	33	1	3.882	4.945	33	1	3.882	4.945
9	0	4.598	5.896	9	0	3.069	4.229	9	0	2.410	3.479	34	0	2.410	3.479	34	0	2.410	3.479
9	1	6.529	7.700	9	1	4.730	5.918	9	1	3.918	5.025	35	0	3.918	5.025	35	0	3.918	5.025
10	0	4.635	5.989	10	0	3.106	4.321	10	0	2.447	3.571	36	0	2.447	3.571	36	0	2.447	3.571
10	1	6.567	7.800	10	1	4.767	6.035	10	1	3.956	5.115	37	0	3.956	5.115	37	0	3.956	5.115
11	0	4.673	6.081	11	0	3.144	4.413	11	0	2.486	3.662	38	0	2.486	3.662	38	0	2.486	3.662
11	1	6.605	7.912	11	1	4.805	6.128	11	1	3.994	5.201	39	0	3.994	5.201	39	0	3.994	5.201
12	0	4.712	6.174	12	0	3.183	4.506	12	0	2.525	3.755	40	0	2.525	3.755	40	0	2.525	3.755
12	1	6.644	8.021	12	1	4.845	6.221	12	1	4.033	5.301	41	0	4.033	5.301	41	0	4.033	5.301
13	0	4.752	6.268	13	0	3.223	4.599	13	0	2.565	3.847	42	0	2.565	3.847	42	0	2.565	3.847
13	1	6.684	8.151	13	1	4.884	6.315	13	1	4.073	5.394	43	0	4.073	5.394	43	0	4.073	5.394
14	0	4.792	6.362	14	0	3.264	4.692	14	0	2.605	3.940	44	0	2.605	3.940	44	0	2.605	3.940
14	1	6.724	8.284	14	1	4.925	6.439	14	1	4.114	5.481	45	0	4.114	5.481	45	0	4.114	5.481
15	0	4.833	6.456	15	0	3.304	4.739	15	0	2.646	4.034	46	0	2.646	4.034	46	0	2.646	4.034
15	1	6.765	8.409	15	1	4.966	6.503	15	1	4.154	5.571	47	0	4.154	5.571	47	0	4.154	5.571
16	0	4.875	6.550	16	0	3.346	4.880	16	0	2.687	4.081	48	0	2.687	4.081	48	0	2.687	4.081
16	1	6.807	8.531	16	1	5.007	6.597	16	1	4.196	5.662	49	0	4.196	5.662	49	0	4.196	5.662
17	0	4.916	6.644	17	0	3.388	4.974	17	0	2.729	4.221	50	0	2.729	4.221	50	0	2.729	4.221
17	1	6.849	8.662	17	1	5.049	6.692	17	1	4.238	5.749	51	0	4.238	5.749	51	0	4.238	5.749
18	0	4.959	6.739	18	0	3.430	5.068	18	0	2.771	4.315	52	0	2.771	4.315	52	0	2.771	4.315
18	1	6.891	8.786	18	1	5.091	6.804	18	1	4.280	5.832	53	0	4.280	5.832	53	0	4.280	5.832
19	0	5.001	6.834	19	0	3.472	5.115	19	0	2.814	4.410	54	0	2.814	4.410	54	0	2.814	4.410
19	1	6.933	8.911	19	1	5.134	6.911	19	1	4.323	5.911	55	0	4.323	5.911	55	0	4.323	5.911
20	0	5.044	6.929	20	0	3.515	5.257	20	0	2.856	4.504	56	0	2.856	4.504	56	0	2.856	4.504
20	1	6.976	9.044	20	1	5.177	7.044	20	1	4.366	6.044	57	0	4.366	6.044	57	0	4.366	6.044
21	0	5.087	7.024	21	0	3.558	5.305	21	0	2.900	4.599	58	0	2.900	4.599	58	0	2.900	4.599
21	1	7.019	9.169	21	1	5.220	7.169	21	1	4.409	6.169	59	0	4.409	6.169	59	0	4.409	6.169
22	0	5.130	7.120	22	0	3.601	5.448	22	0	2.943	4.694	60	0	2.943	4.694	60	0	2.943	4.694
22	1	7.063	9.303	22	1	5.264	7.303	22	1	4.453	6.303	61	0	4.453	6.303	61	0	4.453	6.303
23	0	5.174	7.215	23	0	3.645	5.543	23	0	2.987	4.789	62	0	2.987	4.789	62	0	2.987	4.789
23	1	7.107	9.411	23	1	5.307	7.411	23	1	4.496	6.411	63	0	4.496	6.411	63	0	4.496	6.411
24	0	5.218	7.311	24	0	3.689	5.639	24	0	3.031	4.885	64	0	3.031	4.885	64	0	3.031	4.885
24	1	7.151	9.511	24	1	5.351	7.511	24	1	4.540	6.511	65	0	4.540	6.511	65	0	4.540	6.511

Table 1. (Continued)

a1 = 0.1    alpha = 0.01					a1 = 0.1    alpha = 0.05					a1 = 0.1    alpha = 0.10				
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	4.145	17	0	5.384	0	0	2.696	16	1	5.426	0	0	2.072
0	1	5.975	17	1	7.221	0	1	4.269	16	2	6.827	0	1	3.501
0	2	7.565	18	0	5.477	0	2	5.666	17	0	3.936	0	2	4.790
1	0	4.151	18	1	7.314	0	3	6.978	17	1	5.519	0	3	6.013
1	1	5.981	19	0	5.569	1	0	2.702	17	2	6.920	0	4	4.659
1	2	7.571	19	1	7.407	1	1	4.275	18	0	4.028	1	0	2.078
2	0	4.184	20	0	5.662	1	2	5.672	18	1	5.612	1	1	3.507
2	1	6.014	20	1	7.501	2	0	2.735	19	0	4.121	1	2	4.796
2	2	7.605	21	0	5.756	2	1	4.309	19	1	5.706	2	0	2.111
3	0	4.235	21	1	7.596	2	2	5.706	20	0	4.214	2	0	3.501
3	1	6.065	22	0	5.850	3	0	2.787	20	1	5.800	2	1	3.501
3	2	7.656	22	1	7.691	3	1	4.360	21	0	4.307	2	2	4.829
4	0	4.297	23	0	5.945	3	2	5.757	21	1	5.895	2	3	6.052
4	1	6.127	23	1	7.786	4	0	2.849	22	0	4.402	3	0	2.163
4	2	7.718	24	0	6.040	4	1	4.422	22	1	5.991	3	1	3.590
5	0	4.366	24	1	7.882	4	2	5.819	23	0	4.496	3	2	4.881
5	1	6.196	25	0	6.135	5	0	2.917	23	1	6.087	4	0	2.225
5	2	7.786	25	1	7.978	5	1	4.911	24	0	4.591	4	1	3.654
6	0	4.439	26	0	6.230	5	2	5.889	24	1	6.183	4	2	4.943
6	1	6.270	26	1	8.075	6	0	2.991	25	0	4.687	5	0	2.293
6	2	7.861	27	0	6.326	6	1	4.565	25	1	6.280	5	1	3.723
7	0	4.516	27	1	8.172	6	2	5.963	26	0	4.782	6	0	2.367
7	1	6.348	28	0	6.422	7	0	3.068	26	1	6.380	6	1	3.797
7	2	7.939	28	1	8.270	7	1	4.643	27	0	4.878	7	0	2.444
8	0	4.596	29	0	6.519	7	2	6.040	27	1	6.478	7	1	3.875
8	1	6.428	29	1	8.369	8	0	3.148	28	0	4.975	8	0	2.507
8	2	8.020	30	0	6.616	8	1	4.723	28	1	6.577	8	1	3.675
9	0	4.679	30	1	8.467	8	2	6.121	29	0	5.073	9	0	2.606
9	1	6.510	31	0	6.713	9	0	3.230	29	1	6.676	9	1	4.038
9	2	8.102	31	1	8.565	9	1	4.806	30	0	5.170	10	0	2.524
10	0	4.763	32	0	6.810	9	2	6.204	30	1	6.776	10	1	3.955
10	1	6.595	32	1	8.665	10	0	3.314	31	0	5.268	10	2	5.246
10	2	8.187	33	0	6.908	10	1	4.891	31	1	6.878	11	0	2.606
11	0	4.681	34	0	7.006	10	2	6.289	32	0	5.366	11	0	2.690
11	1	6.681	35	0	7.104	11	0	3.400	33	0	5.464	11	1	4.123
12	0	4.935	36	0	7.202	11	1	4.977	34	0	5.563	12	0	2.590
12	1	6.768	37	0	7.300	11	2	6.376	35	0	5.661	12	1	3.951
12	2	8.361	38	0	7.399	12	0	3.487	36	0	5.760	13	0	2.776
13	0	5.023	39	0	7.498	12	1	5.065	37	0	5.859	13	1	4.209
13	1	6.857	40	0	7.600	12	2	6.464	38	0	5.959	14	0	2.863
13	2	8.450	41	0	7.700	13	0	3.575	39	0	6.058	14	1	4.297
14	0	5.112	42	0	7.799	13	1	5.154	40	0	6.158	15	0	2.590
14	1	6.947	43	0	7.899	13	2	6.553	41	0	6.258	16	0	3.951
14	2	8.540	44	0	8.000	14	0	3.664	42	0	6.359	17	0	2.951
15	0	5.202	45	0	8.100	14	1	5.264	43	0	6.459	18	0	2.863
15	1	7.037	46	0	8.200	14	2	6.644	44	0	6.559	19	0	3.040
15	2	8.631	47	0	8.301	15	0	3.754	45	0	6.660	20	0	4.476
16	0	5.293	48	0	8.402	15	1	5.334	46	0	6.761	21	0	3.130
16	1	7.129	49	0	8.503	16	0	6.735	47	0	6.862	22	0	2.590
16	2	8.712	50	0	8.604	16	1	7.844	48	0	6.963	23	0	3.951
17	0	5.384	51	0	8.705	17	0	7.913	49	0	7.064	24	0	4.476
17	1	7.221	52	0	8.806	17	1	8.024	50	0	7.165	25	0	5.246
18	0	5.477	53	0	8.907	18	0	8.135	51	0	7.266	26	0	6.013
18	1	7.314	54	0	9.008	18	1	8.246	52	0	7.367	27	0	6.776
19	0	5.569	55	0	9.109	19	0	8.357	53	0	7.468	28	0	7.535
19	1	7.407	56	0	9.210	19	1	8.468	54	0	7.569	29	0	8.294
20	0	5.662	57	0	9.311	20	0	8.569	55	0	7.670	30	0	9.053
20	1	7.501	58	0	9.412	20	1	8.670	56	0	7.771	31	0	9.812
21	0	5.756	59	0	9.513	21	0	8.771	57	0	7.872	32	0	10.571
21	1	7.596	60	0	9.614	21	1	8.872	58	0	7.973	33	0	11.330
22	0	5.850	61	0	9.715	22	0	8.973	59	0	8.074	34	0	12.089
22	1	7.691	62	0	9.816	22	1	9.074	60	0	8.175	35	0	12.848
23	0	5.945	63	0	9.917	23	0	9.175	61	0	8.276	36	0	13.607
23	1	7.786	64	0	10.018	23	1	9.276	62	0	8.377	37	0	14.366
24	0	6.040	65	0	10.119	24	0	9.377	63	0	8.478	38	0	15.125
24	1	7.882	66	0	10.220	24	1	9.478	64	0	8.579	39	0	15.884
25	0	6.135	67	0	10.321	25	0	9.579	65	0	8.680	40	0	16.643
25	1	7.978	68	0	10.422	25	1	9.680	66	0	8.781	41	0	17.402
26	0	6.230	69	0	10.523	26	0	9.781	67	0	8.882	42	0	18.161
26	1	8.075	70	0	10.624	26	1	9.882	68	0	8.983	43	0	18.920
27	0	6.326	71	0	10.725	27	0	9.983	69	0	9.084	44	0	19.679
27	1	8.172	72	0	10.826	27	1	10.084	70	0	9.185	45	0	20.438
28	0	6.422	73	0	10.927	28	0	10.185	71	0	9.286	46	0	21.197
28	1	8.270	74	0	11.028	28	1	10.286	72	0	9.387	47	0	21.956
29	0	6.519	75	0	11.129	29	0	10.387	73	0	9.488	48	0	22.715
29	1	8.369	76	0	11.230	29	1	10.488	74	0	9.589	49	0	23.474
30	0	6.616	77	0	11.331	30	0	10.589	75	0	9.690	50	0	24.233
30	1	8.467	78	0	11.432	30	1	10.690	76	0	9.791	51	0	24.992
31	0	6.713	79	0	11.533	31	0	10.791	77	0	9.892	52	0	25.751
31	1	8.565	80	0	11.634	31	1	10.892	78	0	9.993	53	0	26.510
32	0	6.810	81	0	11.735	32	0	10.993	79	0	10.094	54	0	27.269
32	1	8.665	82	0	11.836	32	1	11.094	80	0	10.195	55	0	28.028
33	0	6.908	83	0	11.937	33	0	11.195	81	0	10.296	56	0	28.787
34	0	7.006	84	0	12.038	34	0	11.296	82	0	10.397	57	0	29.546
35	0	7.104	85	0	12.139	35	0	11.397	83	0	10.498	58	0	30.305
36	0	7.202	86	0	12.240	36	0	11.498	84	0	10.599	59	0	31.064
37	0	7.300	87	0	12.341	37	0	11.599	85	0	10.700	60	0	31.823
38	0	7.399	88	0	12.442	38	0	11.700	86	0	10.801	61	0	32.582
39	0	7.498	89	0	12.543	39	0	11.801	87	0	10.902	62	0	33.341
40	0	7.600	90	0	12.644	40	0	11.902	88	0	11.003	63	0	34.100
41	0	7.700	91	0	12.745	41	0	12.003	89	0	11.104	64	0	34.859
42	0	7.799	92	0	12.846	42	0	12.104	90	0	11.205	65	0	35.618
43	0	7.899	93	0	12.947	43	0	12.205	91	0	11.306	66	0	36.377
44	0	8.000	94	0	13.048	44	0	12.306	92	0	11.407	67	0	37.136
45	0	8.100	95	0	13.149	45	0	12.407	93	0	11.508	68	0	37.895
46	0	8.200	96	0	13.250	46	0	12.508	94	0	11.609	69	0	38.654
47	0	8.301	97	0	13.351	47	0	12.609	95	0	11.710	70	0	39.413
48	0	8.402	98	0	13.452	48	0	12.710	96	0	11.811	71	0	40.172
49	0	8.503	99	0	13.553	49	0	12.811	97	0	11.912	72	0	40.931
50	0	8.604	100	0	13.654	50	0	12.912	98	0	12.013	73	0	41.690

Table 1. (Continued)

a1 = 0.15				alpha = 0.01				a1 = 0.15				alpha = 0.05				a1 = 0.15				alpha = 0.10			
x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)
0	0	3.914	7.143	13	1	7.143	0	0	2.546	12	0	3.883	0	0	1.957	11	2	5.755					
0	1	5.643	8.659	13	2	8.659	0	1	4.032	12	1	5.403	0	1	3.306	11	3	6.921					
0	2	7.145	5.538	14	0	5.538	0	2	5.351	12	2	6.728	0	2	4.524	12	0	3.293					
0	3	8.538	7.291	14	1	7.291	0	3	6.591	13	0	4.026	0	3	5.679	12	1	4.679					
1	0	3.929	8.809	14	2	8.809	0	4	7.780	13	1	5.560	0	4	6.795	12	2	5.905					
1	1	5.658	5.684	15	0	5.684	1	0	2.561	13	2	6.881	0	0	1.972	13	0	3.438					
1	2	7.160	7.443	15	1	7.443	1	1	4.047	14	0	4.171	1	1	3.321	13	1	4.836					
1	3	8.553	8.964	15	2	8.964	1	2	5.367	14	1	5.717	1	2	4.530	13	2	6.055					
2	0	3.994	5.831	16	0	5.831	1	3	6.606	14	2	7.034	1	3	5.694	14	0	3.583					
2	1	5.723	7.594	16	1	7.594	1	4	7.796	15	0	4.319	1	4	6.810	14	1	4.993					
2	2	7.225	9.117	16	2	9.117	2	0	2.626	15	1	5.874	2	0	2.037	14	2	6.206					
2	3	8.619	5.979	17	0	5.979	2	1	4.113	15	2	7.187	2	1	3.387	15	0	3.735					
3	0	4.087	7.747	17	1	7.747	2	2	5.432	16	0	4.468	2	2	4.605	15	1	5.150					
3	1	5.816	9.272	17	2	9.272	2	3	6.671	16	1	6.032	2	3	5.759	15	2	6.368					
3	2	7.319	6.128	18	0	6.128	3	0	2.719	16	2	7.341	2	4	6.875	15	2	6.884					
3	3	8.712	7.901	18	1	7.901	3	1	4.206	17	0	4.617	3	0	2.129	16	1	3.307					
4	0	4.194	9.427	18	2	9.427	3	2	5.525	17	1	6.190	3	1	3.480	16	2	6.522					
4	1	5.924	6.278	19	0	6.278	3	3	6.765	17	2	7.496	3	2	4.698	17	0	4.035					
4	2	7.427	8.056	19	1	8.056	4	0	2.826	18	0	4.768	3	3	5.854	17	1	5.464					
4	3	8.821	9.583	19	2	9.583	4	1	4.314	18	1	6.348	3	4	6.970	17	2	6.676					
5	0	4.311	6.429	20	0	6.429	4	2	5.635	18	2	7.651	4	0	2.237	18	0	4.186					
5	1	6.042	8.212	20	1	8.212	4	3	6.875	19	0	4.919	4	1	3.589	18	1	5.623					
5	2	7.546	6.588	21	0	6.588	5	0	2.943	19	1	6.506	4	2	4.808	18	2	6.830					
5	3	8.941	8.381	21	1	8.381	5	1	4.433	20	0	5.072	5	0	2.354	19	0	4.339					
6	0	4.435	6.741	22	0	6.741	5	2	5.754	20	1	6.665	5	1	3.708	19	1	5.780					
6	1	6.167	8.538	22	1	8.538	6	0	3.067	21	0	5.225	5	2	4.928	20	0	4.491					
6	2	7.672	6.896	23	0	6.896	6	1	4.559	21	1	6.824	6	1	3.834	21	0	4.645					
6	3	9.067	8.696	23	1	8.696	6	2	5.881	22	0	5.378	6	2	5.056	22	0	4.799					
7	0	4.564	7.050	24	0	7.050	6	3	7.123	22	1	6.982	6	3	6.216	22	1	6.254					
7	1	6.297	8.855	24	1	8.855	7	0	3.196	23	0	5.532	7	1	3.966	23	0	4.954					
7	2	7.803	7.206	25	0	7.206	7	1	4.690	23	1	7.141	7	2	5.189	24	0	5.109					
7	3	9.199	9.014	25	1	9.014	7	2	6.013	24	0	5.687	7	3	6.347	24	1	6.569					
8	0	4.696	7.362	26	0	7.362	7	3	7.256	24	1	7.300	7	4	5.843	25	0	5.264					
8	1	6.435	9.173	26	1	9.173	8	0	3.328	25	0	5.843	7	5	6.485	25	1	6.727					
8	2	7.939	7.519	27	0	7.519	8	1	4.825	25	1	7.459	8	0	4.102	26	0	5.419					
8	3	9.336	9.333	27	1	9.333	8	2	6.150	26	0	5.998	8	1	5.327	26	1	6.885					
9	0	4.831	7.675	28	0	7.675	8	3	7.394	26	1	7.617	8	2	6.154	27	0	5.575					
9	1	6.569	9.492	28	1	9.492	9	0	3.463	27	0	6.310	8	3	6.485	27	1	6.887					
9	2	8.078	7.833	29	0	7.833	9	1	4.984	27	1	7.776	9	0	2.874	28	0	5.731					
9	3	9.476	7.991	30	0	7.991	9	2	6.290	28	0	6.467	9	1	4.241	28	1	6.201					
10	0	4.969	8.148	31	0	8.148	9	3	7.536	29	0	6.624	9	2	5.469	29	0	6.044					
10	1	6.709	8.307	32	0	8.307	10	0	3.601	30	0	6.781	9	3	6.627	30	0	6.201					
10	2	8.223	8.465	33	0	8.465	10	1	5.105	31	0	7.096	10	1	4.384	31	0	6.357					
11	0	5.109	8.624	34	0	8.624	10	2	6.681	32	0	7.253	10	2	5.610	32	0	6.514					
11	1	6.852	8.783	35	0	8.783	11	0	3.741	33	0	7.411	11	0	6.772	33	0	6.671					
11	2	8.364	8.942	36	0	8.942	11	1	5.249	34	0	7.569	11	1	4.529	34	0	6.828					
12	0	5.251	9.101	37	0	9.101	11	2	6.579	35	0	7.727	11	2	7.727	35	0	6.986					
12	1	6.997	9.260	38	0	9.260	11	3	7.829	36	0	7.727	11	3	7.829	36	0	6.986					
12	2	8.510	9.420	39	0	9.420																	
13	0	5.394	9.580	40	0	9.580																	

Table 1. (Continued)

a1 = 0.2    alpha = 0.01				a1 = 0.2    alpha = 0.05				a1 = 0.2    alpha = 0.10			
x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)
0	0	3.684	8.377	0	0	2.397	7.664	0	0	1.842	4.572
0	1	5.311	9.716	0	1	3.795	3.982	0	1	3.112	5.716
0	2	6.725	5.469	0	2	5.037	10 1 5.440	0	2	4.258	6.820
0	3	8.036	7.153	0	3	6.203	10 2 6.697	0	3	5.345	10 0 3.438
0	4	9.284	11 1 7.590	0	4	7.323	10 3 7.873	0	4	6.395	10 1 4.786
1	0	3.714	9.938	0	5	8.410	11 0 4.187	0	5	7.420	10 2 5.925
1	1	5.341	12 0 5.676	1	0	2.427	11 1 5.652	1	0	1.872	10 3 7.030
1	2	6.755	12 1 7.381	1	1	3.825	11 2 6.908	1	1	3.142	11 0 3.646
1	3	8.067	12 2 8.805	1	2	5.067	11 3 8.084	1	2	4.288	11 1 5.001
1	4	9.314	12 3 10.155	1	3	6.233	12 0 4.395	1	3	5.375	11 2 6.135
2	0	3.821	13 0 5.885	1	4	7.353	12 1 5.665	1	4	6.425	11 3 7.261
2	1	5.449	13 1 7.598	1	5	8.441	12 2 7.120	1	5	7.450	12 0 3.856
2	2	6.864	13 2 9.020	2	0	2.534	12 3 8.295	2	0	1.979	12 1 5.215
2	3	8.176	14 0 6.096	2	1	3.934	13 0 4.605	2	1	3.251	12 2 6.345
2	4	9.424	14 1 7.815	2	2	5.176	13 1 6.158	2	2	4.398	12 3 7.471
3	0	3.965	14 2 9.236	2	3	6.343	13 2 7.332	2	3	5.486	13 0 4.067
3	1	5.595	15 0 6.308	2	4	7.464	13 3 8.506	2	4	6.536	13 1 5.430
3	2	7.011	15 1 8.034	2	5	8.552	14 0 4.816	2	5	7.561	13 2 6.555
3	3	8.324	15 2 9.452	3	0	2.677	14 1 6.320	3	0	2.123	13 3 7.681
3	4	9.573	16 0 6.522	3	1	4.081	14 2 7.544	3	1	3.399	14 0 4.279
4	0	4.128	16 1 8.252	3	2	5.325	15 0 5.072	3	2	4.547	14 1 5.644
4	1	5.761	16 2 9.669	3	3	6.493	15 1 6.536	3	3	5.636	14 2 6.766
4	2	7.179	17 0 6.776	3	4	7.614	15 2 7.757	3	4	6.688	15 0 4.491
4	3	8.494	17 1 8.471	4	0	4.248	16 0 5.288	3	5	7.714	15 1 5.859
4	4	9.744	17 2 9.845	4	1	4.248	16 1 6.752	4	0	2.286	15 2 6.930
5	0	4.302	18 0 6.996	4	2	5.496	16 2 7.970	4	1	3.568	16 0 4.705
5	1	5.940	18 1 8.691	4	3	6.664	17 0 5.506	4	2	4.719	16 1 6.073
5	2	7.361	18 2 10.060	4	4	7.787	17 1 6.967	4	3	5.810	16 2 7.139
5	3	8.678	19 0 7.216	5	0	3.015	17 2 8.183	4	4	6.864	17 0 4.919
5	4	9.926	19 1 8.910	5	1	4.430	18 0 5.723	5	0	2.460	17 1 6.287
6	0	4.485	19 2 10.276	5	2	5.681	18 1 7.183	5	1	3.751	17 2 7.349
6	1	6.129	20 0 7.436	5	3	6.850	18 2 8.397	5	2	4.904	18 0 5.133
6	2	7.552	20 1 9.129	5	4	7.975	19 0 5.941	5	3	5.999	18 1 6.502
6	3	8.872	21 0 7.657	6	0	3.198	19 1 7.399	5	4	7.056	18 2 7.559
6	4	10.118	21 1 9.349	6	1	4.640	20 0 6.104	6	0	2.643	19 0 5.348
7	0	4.674	22 0 7.877	6	2	5.890	20 1 7.614	6	1	3.944	19 1 6.716
7	1	6.339	22 1 9.569	6	3	7.044	21 0 6.376	6	2	5.099	20 0 5.563
7	2	7.750	23 0 8.098	6	4	8.172	21 1 7.830	6	3	6.198	20 1 6.978
7	3	9.075	23 1 9.886	7	0	3.386	22 0 6.594	6	4	7.228	21 0 5.778
7	4	10.316	24 0 8.319	7	1	4.855	22 1 8.046	7	0	2.832	21 1 7.189
8	0	4.867	24 1 10.103	7	2	6.066	23 0 6.812	7	1	4.146	22 0 5.994
8	1	6.557	25 0 8.540	7	3	7.244	23 1 8.262	7	2	5.299	22 1 7.401
8	2	7.954	26 0 8.761	7	4	8.375	24 0 7.029	7	3	6.402	23 0 6.209
8	3	9.285	27 0 9.982	8	0	3.579	24 1 8.477	7	4	7.422	23 1 7.613
9	0	5.064	28 0 9.203	8	1	5.019	25 0 7.247	8	0	3.025	24 0 6.425
9	1	6.729	29 0 9.425	8	2	6.269	26 0 7.465	8	1	4.356	25 0 6.641
9	2	8.161	30 0 9.646	8	3	7.449	27 0 7.683	8	2	5.509	26 0 6.857
9	3	9.500	31 0 9.802	9	0	3.776	28 0 7.901	8	3	6.612	27 0 7.073
10	0	5.264	32 0 10.023	9	1	5.230	29 0 8.119	8	4	7.627	28 0 7.289
10	1	6.940	33 0 10.244	9	2	6.474	30 0 8.337	9	0	3.224	29 0 7.506

Table 1. (Continued)

a1 = 0.25    alpha = 0.01				a1 = 0.25    alpha = 0.05				a1 = 0.25    alpha = 0.10			
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	3.454	8	4	10.491	0	0	2.247	8	1	5.307
0	1	4.979	9	0	5.406	0	1	3.558	8	2	6.432
0	2	6.304	9	1	7.041	0	2	4.722	8	3	7.992
0	3	7.534	9	2	8.345	0	3	5.815	8	4	8.680
0	4	8.704	9	3	9.581	0	4	6.865	8	5	9.995
0	5	9.831	9	4	10.815	0	5	7.885	9	0	3.716
1	0	3.508	10	0	5.718	0	6	8.882	9	1	4.986
1	1	5.033	10	1	7.325	1	0	2.301	9	2	6.022
1	2	6.359	10	2	8.624	1	1	3.613	9	3	7.028
1	3	7.589	10	3	9.857	1	2	4.777	9	4	8.005
1	4	8.762	11	0	6.003	1	3	5.871	10	0	3.991
1	5	9.887	11	1	7.609	1	4	6.921	10	1	5.261
2	0	3.671	11	2	8.902	1	5	7.941	10	2	6.292
2	1	5.200	11	3	10.133	1	6	8.938	10	3	7.294
2	2	6.528	12	0	6.288	2	0	2.464	11	0	4.268
2	3	7.759	12	1	7.893	2	1	3.781	11	1	5.533
2	4	8.931	12	2	9.183	2	2	4.948	11	2	6.490
2	5	10.060	12	3	10.410	2	3	6.044	11	3	7.559
3	0	3.878	13	0	6.574	2	4	7.096	12	0	4.545
3	1	5.435	13	1	8.178	2	5	8.118	12	1	5.805
3	2	6.758	13	2	9.463	2	6	9.166	12	2	6.756
3	3	7.981	13	3	10.687	3	0	2.671	12	3	7.825
3	4	9.155	14	0	6.861	3	1	4.001	13	0	4.823
3	5	10.288	14	1	8.462	3	2	5.172	13	1	6.077
4	0	4.108	14	2	9.748	3	3	6.274	13	2	7.095
4	1	5.656	15	0	7.148	3	4	7.333	13	3	8.091
4	2	6.988	15	1	8.747	4	0	2.901	14	0	5.101
4	3	8.228	15	2	10.019	4	1	4.267	14	1	6.350
4	4	9.407	16	0	7.435	4	2	5.422	14	2	7.363
4	5	10.545	16	1	9.031	4	3	6.537	14	3	8.230
5	0	4.352	16	2	10.299	4	4	7.599	15	0	5.379
5	1	5.914	17	0	7.723	4	5	8.569	15	1	6.622
5	2	7.244	17	1	9.319	5	0	3.145	15	2	7.712
5	3	8.489	17	2	10.579	5	1	4.549	16	0	5.658
5	4	9.672	18	0	8.010	5	2	5.639	16	1	6.895
5	5	10.754	18	1	9.603	5	3	6.738	16	2	7.952
6	0	4.605	18	2	10.860	5	4	7.793	17	0	5.937
6	1	6.182	19	0	8.298	5	5	8.815	17	1	7.167
6	2	7.507	19	1	9.888	6	0	3.398	18	0	6.216
6	3	8.757	20	0	8.586	6	1	4.755	18	1	7.440
6	4	9.944	20	1	10.173	6	2	5.961	19	0	6.498
7	0	4.864	21	0	8.874	6	3	7.000	19	1	7.590
7	1	6.459	21	1	10.458	6	4	8.052	20	0	6.777
7	2	7.792	22	0	9.162	6	5	9.075	20	1	7.862
7	3	9.036	22	1	10.743	7	0	3.665	21	0	7.055
7	4	10.217	23	0	9.450	7	1	5.035	21	1	8.133
8	0	5.133	24	0	9.738	7	2	6.166	22	0	7.336
8	1	6.738	25	0	10.036	7	3	7.262	23	0	7.682
8	2	8.068	26	0	10.325	7	4	8.312	24	0	7.985
8	3	9.307	27	0	10.613	8	0	3.936	25	0	8.232



Table 1. (Continued)

a1 = 0.3					alpha = 0.01					a1 = 0.3					alpha = 0.05					a1 = 0.3					alpha = 0.10				
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)			
0	0	3.224	7	2	7.907	0	0	2.097	7	1	5.319	0	0	1.612	6	5	8.102	0	0	1.612	6	5	8.102	0	0	1.612			
0	1	4.647	7	3	9.143	0	1	3.321	7	2	6.465	0	1	2.723	7	0	3.581	0	1	2.723	7	0	3.581	0	1	2.723			
0	2	5.884	7	4	10.206	0	2	4.407	7	3	7.427	0	2	3.726	7	1	4.761	0	2	3.726	7	1	4.761	0	2	3.726			
0	3	7.032	7	5	11.097	0	3	5.428	7	4	8.440	0	3	4.676	7	2	5.773	0	3	4.676	7	2	5.773	0	3	4.676			
0	4	8.123	8	0	5.549	0	4	6.407	7	5	9.386	0	4	5.597	7	3	6.539	0	4	5.597	7	3	6.539	0	4	5.597			
0	5	9.176	8	1	7.104	0	5	7.359	8	0	4.446	0	5	6.492	7	4	7.540	0	5	6.492	7	4	7.540	0	5	6.492			
0	6	10.199	8	2	8.257	0	6	8.290	8	1	5.751	0	6	7.373	7	5	8.260	0	6	7.373	7	5	8.260	0	6	7.373			
0	7	11.200	8	3	9.476	0	7	9.204	8	2	6.810	0	7	8.240	8	0	3.929	0	7	8.240	8	0	3.929	0	7	8.240			
1	0	3.315	8	4	10.545	1	0	2.189	8	3	7.667	1	0	1.703	8	1	5.101	1	0	1.703	8	1	5.101	1	0	1.703			
1	1	4.741	9	0	5.910	1	1	3.416	8	4	8.620	1	1	2.818	8	2	6.109	1	1	2.818	8	2	6.109	1	1	2.818			
1	2	5.979	9	1	7.465	1	2	4.504	9	0	4.801	1	2	3.823	8	3	6.965	1	2	3.823	8	3	6.965	1	2	3.823			
1	3	7.128	9	2	8.607	1	3	5.525	9	1	6.006	1	3	4.776	8	4	7.862	1	3	4.776	8	4	7.862	1	3	4.776			
1	4	8.220	9	3	9.816	1	4	6.506	9	2	6.987	1	4	5.696	9	0	4.279	1	4	5.696	9	0	4.279	1	4	5.696			
1	5	9.274	9	4	10.885	1	5	7.459	9	3	8.164	1	5	6.594	9	1	5.284	1	5	6.594	9	1	5.284	1	5	6.594			
1	6	10.298	10	0	6.273	1	6	8.390	9	4	8.953	1	6	7.475	9	2	6.413	1	6	7.475	9	2	6.413	1	6	7.475			
2	0	3.552	10	1	7.727	2	0	2.425	10	0	5.155	2	0	1.940	9	3	7.296	2	0	1.940	9	3	7.296	2	0	1.940			
2	1	4.992	10	2	8.959	2	1	3.666	10	1	6.354	2	1	3.074	10	0	4.629	2	1	3.074	10	0	4.629	2	1	3.074			
2	2	6.232	10	3	9.989	2	2	4.765	10	2	7.487	2	2	4.096	10	1	5.625	2	2	4.096	10	1	5.625	2	2	4.096			
2	3	7.386	10	4	11.248	2	3	5.798	10	3	8.341	2	3	5.060	10	2	6.745	2	3	5.060	10	2	6.745	2	3	5.060			
2	4	8.484	11	0	6.637	2	4	6.785	10	4	9.284	2	4	5.932	10	3	7.612	2	4	5.932	10	3	7.612	2	4	5.932			
2	5	9.543	11	1	8.081	2	5	7.695	11	0	5.510	2	5	6.830	10	4	8.508	2	5	6.830	10	4	8.508	2	5	6.830			
2	6	10.573	11	2	9.311	2	6	8.627	11	1	6.699	2	6	7.711	11	0	4.982	2	6	7.711	11	0	4.982	2	6	7.711			
3	0	3.840	11	3	10.332	2	7	9.305	11	2	7.825	2	7	8.579	11	1	5.950	2	7	8.579	11	1	5.950	2	7	8.579			
3	1	5.310	12	0	7.001	3	0	2.714	11	3	8.772	3	0	2.228	11	2	7.076	3	0	2.228	11	2	7.076	3	0	2.228			
3	2	6.538	12	1	8.435	3	1	3.970	12	0	5.865	3	1	3.384	11	3	7.937	3	1	3.384	11	3	7.937	3	1	3.384			
3	3	7.699	12	2	9.665	3	2	5.082	12	1	7.108	3	2	4.394	12	0	5.399	3	2	4.394	12	0	5.399	3	2	4.394			
3	4	8.804	12	3	10.675	3	3	6.044	12	2	7.999	3	3	5.294	12	1	6.283	3	3	5.294	12	1	6.283	3	3	5.294			
3	5	9.870	13	0	7.367	3	4	7.026	12	3	9.104	3	4	6.212	12	2	7.407	3	4	6.212	12	2	7.407	3	4	6.212			
3	6	10.909	13	1	8.788	3	5	8.065	13	0	6.221	3	5	5.590	13	0	4.784	3	5	5.590	13	0	4.784	3	5	5.590			
4	0	4.154	13	2	10.089	3	6	8.907	13	1	7.289	3	6	7.992	13	1	6.617	3	6	7.992	13	1	6.617	3	6	7.992			
4	1	5.663	13	3	11.018	4	0	3.027	13	2	8.502	4	0	2.542	13	2	7.339	4	0	2.542	13	2	7.339	4	0	2.542			
4	2	6.865	14	0	7.826	4	1	4.294	14	0	6.577	4	1	3.710	13	3	8.184	4	1	3.710	13	3	8.184	4	1	3.710			
4	3	8.028	14	1	9.133	4	2	5.412	14	1	7.629	4	2	4.641	14	0	6.079	4	2	4.641	14	0	6.079	4	2	4.641			
4	4	9.042	14	2	10.438	4	3	6.341	14	2	8.841	4	3	5.590	14	1	6.951	4	3	5.590	14	1	6.951	4	3	5.590			
4	5	10.097	15	0	8.186	4	4	7.322	14	3	9.634	4	4	6.461	14	2	8.068	4	4	6.461	14	2	8.068	4	4	6.461			
4	6	11.212	15	1	9.496	4	5	8.275	15	0	6.934	4	5	5.745	15	0	6.446	4	5	5.745	15	0	6.446	4	5	5.745			
5	0	4.483	15	2	10.785	4	6	9.213	15	1	8.131	4	6	8.425	15	1	7.284	4	6	8.425	15	1	7.284	4	6	8.425			
5	1	6.024	16	0	8.547	5	0	3.360	15	2	9.179	4	6	7.387	15	2	8.399	5	0	3.360	15	2	8.399	5	0	3.360			
5	2	7.214	16	1	9.850	5	1	4.637	16	0	7.397	5	1	4.047	15	3	9.399	5	1	4.047	15	3	9.399	5	1	4.047			
5	3	8.371	16	2	11.131	5	2	5.658	16	1	8.310	5	2	4.962	16	1	7.630	5	2	4.962	16	1	7.630	5	2	4.962			
5	4	9.382	17	0	8.909	5	3	6.670	16	2	9.518	5	3	5.992	16	2	8.573	5	3	5.992	16	2	8.573	5	3	5.992			
5	5	10.433	17	1	10.203	5	4	7.777	17	0	7.735	5	4	6.898	16	3	9.633	5	4	6.898	16	3	9.633	5	4	6.898			
6	0	4.837	18	0	9.270	5	5	8.728	17	1	8.652	5	5	7.787	17	0	7.118	6	0	4.837	18	0	7.118	6	0	4.837			
6	1	6.384	18	1	10.557	6	0	3.742	18	0	8.008	6	1	3.235	17	1	7.963	6	1	3.235	17	1	7.963	6	1	3.235			
6	2	7.559	19	0	9.631	6	1	4.977	19	0	8.993	6	2	3.426	18	1	8.296	6	2	3.426	18	1	8.296	6	2	3.426			
6	3	8.712	19	1	10.911	6	2	6.129	19	1	9.333	6	3	4.427	18	2	9.791	6	3	4.427	18	2	9.791	6	3	4.427			
6	4	9.714	20	0	9.994	6	3	7.148	19	2	10.427	6	4	6.311	19	0	8.128	6	4	6.311	19	0	8.128	6	4	6.311			
6	5	10.763	21	0	10.356	6	4	7.969	20	0	8.770	6	5	7.215	20	0	7.963	6	5	7.215	20	0	7.963	6	5	7.215			
7	0	5.192	22	0	10.718	6	5	9.056	21	0	9.117	6	6	8.462	21	0	8.465	7	0	5.192	22	0	8.465	7	0	5.192			
7	1	6.744	23	0	11.080	7	0	4.094	22	0	9.462	7	1	3.519	22	0	8.465	7	1	3.519	22	0	8.465	7	1	3.519			

Table 1. (Continued)

a1 = 0.35    alpha = 0.01				a1 = 0.35    alpha = 0.05				a1 = 0.35    alpha = 0.10			
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.993	6	2	7.650	0	0	1.947	6	0	3.687
0	1	4.315	6	3	8.678	0	1	3.084	6	1	4.547
0	2	5.464	6	4	9.681	0	2	4.092	6	2	5.536
0	3	6.529	6	5	10.649	0	3	5.040	6	3	6.490
0	4	7.543	6	6	11.583	0	4	5.950	6	4	7.323
0	5	8.520	7	0	5.763	0	5	6.834	6	5	7.950
0	6	9.471	7	1	6.965	0	6	7.698	6	6	8.949
0	7	10.400	7	2	8.082	0	7	8.546	7	0	4.120
0	8	11.312	7	3	9.102	0	8	9.383	7	1	4.947
1	0	3.146	7	4	10.094	1	0	2.100	7	2	5.962
1	1	4.478	7	5	11.059	1	1	3.243	7	3	6.946
1	2	5.672	8	0	6.211	1	2	4.256	7	4	7.761
1	3	6.692	8	1	7.396	1	3	5.207	7	5	8.333
1	4	7.708	8	2	8.510	1	4	6.119	8	0	4.641
1	5	8.687	8	3	9.520	1	5	7.005	8	1	5.348
1	6	9.639	8	4	10.508	1	6	7.871	8	2	6.343
1	7	10.569	8	5	11.299	1	7	8.721	8	3	7.071
1	8	11.482	9	0	6.659	1	8	9.559	8	4	8.096
2	0	3.485	9	1	7.828	2	0	2.439	9	0	5.048
2	1	4.871	9	2	8.937	2	1	3.587	9	1	5.749
2	2	5.971	9	3	9.938	2	2	4.603	9	2	6.749
2	3	7.043	9	4	10.923	2	3	5.557	9	3	7.462
2	4	8.056	10	0	7.108	2	4	6.471	9	4	8.483
2	5	9.037	10	1	8.260	2	5	7.359	10	0	5.454
2	6	9.990	10	2	9.345	2	6	8.123	10	1	6.149
2	7	10.921	10	3	10.356	2	7	8.971	10	2	7.210
3	0	3.881	10	4	11.470	3	0	2.834	10	3	7.852
3	1	5.203	11	0	7.557	3	1	3.971	10	4	8.870
3	2	6.361	11	1	8.721	3	2	4.984	11	0	5.860
3	3	7.440	11	2	9.768	3	3	5.818	11	1	6.548
3	4	8.439	11	3	10.774	3	4	6.724	11	2	7.603
3	5	9.419	12	0	8.007	3	5	7.743	11	3	8.243
3	6	10.373	12	1	9.151	3	6	8.608	11	4	9.224
3	7	11.179	12	2	10.191	3	7	9.312	12	0	6.266
4	0	4.304	12	3	11.191	4	0	3.301	12	1	7.996
4	1	5.615	13	0	8.457	4	1	4.402	12	2	8.533
4	2	6.803	13	1	9.580	4	2	5.399	12	3	9.518
4	3	7.869	13	2	10.613	4	3	6.342	13	0	6.571
4	4	8.859	14	0	8.909	4	4	7.140	13	1	7.344
4	5	9.834	14	1	10.008	4	5	8.241	13	2	8.389
4	6	10.783	14	2	11.034	4	6	9.078	14	0	7.078
5	0	4.758	15	0	9.365	4	7	9.718	14	1	7.736
5	1	6.103	15	1	10.437	5	0	3.736	14	2	8.781
5	2	7.226	15	2	11.437	5	1	4.817	15	0	7.680
5	3	8.291	16	0	9.811	5	2	5.998	15	1	8.137
5	4	9.267	16	1	10.864	5	3	6.860	16	0	7.881
5	5	10.239	17	0	10.258	5	4	7.528	16	1	8.532
5	6	11.374	17	1	11.291	5	5	8.401	17	0	8.282
6	0	5.313	18	0	10.703	5	6	9.460	17	1	8.927
6	1	6.533	19	0	11.146	6	0	4.174	18	0	8.682

Table 1. (Continued)

a1 = 0.4				alpha = 0.01				a1 = 0.4				alpha = 0.05				a1 = 0.4				alpha = 0.10						
x1	x2	t(x1,x2)		x1	x2	t(x1,x2)		x1	x2	t(x1,x2)		x1	x2	t(x1,x2)		x1	x2	t(x1,x2)		x1	x2	t(x1,x2)				
0	0	2.763	5	4	9.059		0	0	1.797	5	3	6.770	0	0	1.382	5	3	6.053		0	0	1.382	5	3	6.053	
0	1	3.983	5	5	9.926		0	1	2.846	5	4	7.580	0	1	2.334	5	4	6.792		0	1	2.334	5	4	6.792	
0	2	5.044	5	6	10.889		0	2	3.778	5	5	8.373	0	2	3.193	5	5	7.349		0	2	3.193	5	5	7.349	
0	3	6.027	6	0	5.902		0	3	4.652	5	6	9.153	0	3	4.008	5	6	8.414		0	3	4.008	5	6	8.414	
0	4	6.963	6	1	6.748		0	4	5.492	5	7	9.913	0	4	4.796	5	7	8.821		0	4	4.796	5	7	8.821	
0	5	7.865	6	2	7.943		0	5	6.308	6	0	4.737	0	5	5.565	6	0	4.213		0	5	5.565	6	0	4.213	
0	6	8.742	6	3	8.651		0	6	7.114	6	1	5.576	0	6	6.319	6	1	4.951		0	6	6.319	6	1	4.951	
0	7	9.600	6	4	9.763		0	7	7.889	6	2	6.421	0	7	7.063	6	2	5.739		0	7	7.063	6	2	5.739	
0	8	10.442	6	5	10.521		0	8	8.661	6	3	7.241	0	8	7.797	6	3	6.305		0	8	7.797	6	3	6.305	
0	9	11.270	6	6	11.370		0	9	9.423	6	4	8.047	0	9	8.524	6	4	7.564		0	9	8.524	6	4	7.564	
1	0	3.022	7	0	6.442		1	0	2.057	6	5	8.836	1	0	1.641	6	5	8.016		1	0	1.641	6	5	8.016	
1	1	4.270	7	1	7.258		1	1	3.080	6	6	9.577	1	1	2.563	6	6	9.033		1	1	2.563	6	6	9.033	
1	2	5.368	7	2	8.456		1	2	4.103	7	0	5.259	1	2	3.416	7	0	4.708		1	2	3.416	7	0	4.708	
1	3	6.258	7	3	9.370		1	3	4.876	7	1	6.065	1	3	4.228	7	1	5.427		1	3	4.228	7	1	5.427	
1	4	7.190	7	4	10.260		1	4	5.713	7	2	6.899	1	4	5.014	7	2	6.446		1	4	5.014	7	2	6.446	
1	5	8.090	7	5	11.006		1	5	6.528	7	3	7.712	1	5	5.782	7	3	6.973		1	5	5.782	7	3	6.973	
1	6	8.966	7	6	11.703		1	6	7.324	7	4	8.523	1	6	6.535	7	4	7.796		1	6	6.535	7	4	7.796	
1	7	9.822	8	0	6.980		1	7	8.107	7	5	9.301	1	7	7.277	7	5	8.251		1	7	7.277	7	5	8.251	
1	8	10.663	8	1	7.797		1	8	8.878	8	0	5.774	1	8	8.011	8	0	5.198		1	8	8.011	8	0	5.198	
1	9	11.491	8	2	8.967		1	9	9.640	8	1	6.552	1	9	8.737	8	1	5.891		1	9	8.737	8	1	5.891	
2	0	3.514	8	3	9.646		2	0	2.549	8	2	7.377	2	0	2.133	8	2	6.833		2	0	2.133	8	2	6.833	
2	1	4.784	8	4	10.635		2	1	3.503	8	3	8.183	2	1	2.974	8	3	7.216		2	1	2.974	8	3	7.216	
2	2	5.843	8	5	11.490		2	2	4.415	8	4	8.983	2	2	3.818	8	4	8.467		2	2	3.818	8	4	8.467	
2	3	6.675	9	0	7.519		2	3	5.282	8	5	9.751	2	3	4.626	8	5	8.702		2	3	4.626	8	5	8.702	
2	4	7.604	9	1	8.317		2	4	6.118	9	0	6.282	2	4	5.409	9	0	5.682		2	4	5.409	9	0	5.682	
2	5	8.502	9	2	9.248		2	5	6.932	9	1	7.037	2	5	6.175	9	1	6.148		2	5	6.175	9	1	6.148	
2	6	9.376	9	3	10.145		2	6	7.728	9	2	7.854	2	6	6.927	9	2	7.386		2	6	6.927	9	2	7.386	
2	7	10.227	9	4	11.126		2	7	8.502	9	3	8.653	2	7	7.668	9	3	7.674		2	7	7.668	9	3	7.674	
2	8	11.072	10	0	8.058		2	8	9.272	9	4	9.615	2	8	8.358	9	4	8.878		2	8	9.127	9	4	8.878	
3	0	4.077	10	1	8.835		3	0	10.033	10	0	6.785	3	0	2.681	10	0	6.282		3	0	2.681	10	0	6.282	
3	1	5.169	10	2	9.986		3	1	3.158	10	1	7.521	3	1	3.532	10	1	6.615		3	1	3.532	10	1	6.615	
3	2	6.394	10	3	10.751		3	2	3.982	10	2	8.330	3	2	4.352	10	2	7.889		3	2	4.352	10	2	7.889	
3	3	7.287	10	4	11.784		3	3	5.086	10	3	9.123	3	3	5.144	10	3	8.131		3	3	5.144	10	3	8.131	
3	4	8.193	11	0	8.596		3	4	5.831	10	4	9.985	3	4	5.917	11	0	6.752		3	4	5.917	11	0	6.752	
3	5	9.169	11	1	9.476		3	5	6.651	11	0	7.283	3	5	6.672	11	1	7.080		3	5	6.672	11	1	7.080	
3	6	10.038	11	2	10.370		3	6	7.452	11	1	8.002	3	6	7.485	11	2	8.301		3	6	7.485	11	2	8.301	
3	7	10.780	11	3	11.248		3	7	8.238	11	2	9.397	3	7	8.154	11	3	8.587		3	7	8.154	11	3	8.587	
3	8	11.611	12	0	9.128		3	8	9.013	11	3	9.777	3	8	8.948	12	0	7.236		3	8	8.948	12	0	7.236	
4	0	4.655	12	1	9.863		4	0	9.778	12	0	7.777	4	0	3.197	12	1	7.539		4	0	3.197	12	1	7.539	
4	1	5.687	12	2	10.868		4	1	3.679	12	1	8.482	4	1	3.996	12	2	8.758		4	1	3.996	12	2	8.758	
4	2	6.909	12	3	11.634		4	2	4.581	12	2	9.524	4	2	4.814	12	3	9.042		4	2	4.814	12	3	9.042	
4	3	7.760	13	0	9.656		4	3	5.450	12	3	9.868	4	3	5.598	13	0	7.701		4	3	5.598	13	0	7.701	
4	4	8.773	13	1	10.489		4	4	6.289	13	0	8.267	4	4	6.510	13	1	8.005		4	4	6.510	13	1	8.005	
4	5	9.539	13	2	11.365		4	5	7.104	13	1	8.960	4	5	7.117	13	2	9.213		4	5	7.117	13	2	9.213	
4	6	10.397	14	0	10.178		4	6	7.911	13	2	10.059	4	6	7.865	14	0	8.163		4	6	7.865	14	0	8.163	
4	7	11.239	14	1	10.992		4	7	8.694	14	0	8.755	4	7	8.593	14	1	8.460		4	7	8.593	14	1	8.460	
5	0	5.325	15	0	10.697		5	0	4.209	14	1	9.229	5	0	3.710	15	0	8.925		5	0	3.710	15	0	8.925	
5	1	6.210	15	1	11.497		5	1	4.971	15	1	9.357	5	1	4.478	15	1	9.084		5	1	4.478	15	1	9.084	
5	2	7.428	16	0	11.212		5	2	5.942	15	2	9.703	5	2	5.276	16	0				5	2	5.276	16	0	
5	3	8.277	17	0	11.724		5	3		16	0	9.833	5	3		16	0				5	3		16	0	

Table 1. (Continued)

a1 = 0.45				alpha = 0.01				a1 = 0.45				alpha = 0.05				a1 = 0.45				alpha = 0.10				
x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	x1	x2	t(x1,x2)	t(x1,x2)	
0	0	2.533	7	10.825	4	7	10.825	0	0	1.648	6	8.441	4	6	8.441	0	0	1.266	4	6	7.795	4	6	7.795
0	1	3.651	4	11.574	4	7	11.574	0	1	2.609	4	9.139	4	7	9.139	0	1	2.139	4	7	8.490	4	7	8.490
0	2	4.623	5	0	5.899	4	8	9.829	4	8	9.829	4	8	9.829	4	8	9.829	4	8	2.964	4	8	2.964	
0	3	5.525	5	1	6.666	4	9	10.579	4	9	10.579	4	9	10.579	4	9	10.579	4	9	3.674	5	0	4.233	
0	4	6.382	5	2	7.491	4	10	11.322	4	10	11.322	4	10	11.322	4	10	11.322	4	10	4.419	5	1	4.540	
0	5	7.217	5	3	8.300	4	11	12.071	4	11	12.071	4	11	12.071	4	11	12.071	4	11	5.101	5	2	5.470	
0	6	8.014	5	4	9.107	4	12	12.822	4	12	12.822	4	12	12.822	4	12	12.822	4	12	5.793	5	3	6.152	
0	7	8.800	5	5	9.870	4	13	13.571	4	13	13.571	4	13	13.571	4	13	13.571	4	13	6.474	5	4	6.856	
0	8	9.571	5	6	10.634	4	14	14.322	4	14	14.322	4	14	14.322	4	14	14.322	4	14	7.147	5	5	7.505	
0	9	10.331	5	7	11.387	4	15	15.071	4	15	15.071	4	15	15.071	4	15	15.071	4	15	7.813	5	6	8.173	
0	10	11.092	6	0	6.557	5	7	11.387	5	7	11.387	5	7	11.387	5	7	11.387	5	7	8.474	5	7	8.474	
0	11	11.833	6	1	7.129	5	8	12.138	5	8	12.138	5	8	12.138	5	8	12.138	5	8	9.129	6	0	4.791	
1	0	3.021	6	2	8.099	5	9	12.889	5	9	12.889	5	9	12.889	5	9	12.889	5	9	1.750	6	1	5.198	
1	1	4.073	6	3	8.894	5	10	13.639	5	10	13.639	5	10	13.639	5	10	13.639	5	10	2.528	6	2	6.010	
1	2	5.020	6	4	9.693	5	11	14.389	5	11	14.389	5	11	14.389	5	11	14.389	5	11	3.290	6	3	6.685	
1	3	5.908	6	5	10.444	5	12	15.139	5	12	15.139	5	12	15.139	5	12	15.139	5	12	4.024	6	4	7.385	
1	4	6.757	6	6	11.180	5	13	15.889	5	13	15.889	5	13	15.889	5	13	15.889	5	13	4.742	6	5	8.027	
1	5	7.578	6	7	11.929	5	14	16.639	5	14	16.639	5	14	16.639	5	14	16.639	5	14	5.437	6	6	8.691	
1	6	8.377	7	0	7.288	6	1	7.288	6	1	7.288	6	1	7.288	6	1	7.288	6	1	6.124	7	0	5.344	
1	7	9.160	7	1	7.803	6	2	8.011	6	2	8.011	6	2	8.011	6	2	8.011	6	2	6.802	7	1	5.703	
1	8	9.928	7	2	8.721	6	3	8.894	6	3	8.894	6	3	8.894	6	3	8.894	6	3	7.472	7	2	6.547	
1	9	10.685	7	3	9.506	6	4	9.693	6	4	9.693	6	4	9.693	6	4	9.693	6	4	8.136	7	3	7.055	
1	10	11.432	7	4	10.286	6	5	10.444	6	5	10.444	6	5	10.444	6	5	10.444	6	5	8.794	7	4	7.913	
2	0	3.783	7	5	11.016	6	6	11.180	6	6	11.180	6	6	11.180	6	6	11.180	6	6	2.395	7	5	8.391	
2	1	4.725	7	6	11.773	6	7	11.929	6	7	11.929	6	7	11.929	6	7	11.929	6	7	2.887	7	6	9.207	
2	2	5.633	8	0	7.916	7	0	7.916	7	0	7.916	7	0	7.916	7	0	7.916	7	0	3.830	8	0	5.891	
2	3	6.496	8	1	8.410	7	1	8.410	7	1	8.410	7	1	8.410	7	1	8.410	7	1	4.346	8	1	6.279	
2	4	7.141	8	2	9.312	7	2	9.312	7	2	9.312	7	2	9.312	7	2	9.312	7	2	5.260	8	2	7.215	
2	5	8.145	8	3	10.090	7	3	10.090	7	3	10.090	7	3	10.090	7	3	10.090	7	3	5.940	8	3	7.736	
2	6	8.936	8	4	10.866	7	4	10.866	7	4	10.866	7	4	10.866	7	4	10.866	7	4	6.622	8	4	8.547	
2	7	9.712	8	5	11.613	7	5	11.613	7	5	11.613	7	5	11.613	7	5	11.613	7	5	7.295	8	5	8.943	
2	8	10.476	9	0	8.530	8	0	8.530	8	0	8.530	8	0	8.530	8	0	8.530	8	0	7.961	9	0	6.432	
2	9	11.021	9	1	9.011	8	1	9.011	8	1	9.011	8	1	9.011	8	1	9.011	8	1	8.622	9	1	7.614	
2	10	11.759	9	2	9.902	8	2	9.902	8	2	9.902	8	2	9.902	8	2	9.902	8	2	9.277	9	2	7.611	
3	0	4.520	9	3	10.672	8	3	10.672	8	3	10.672	8	3	10.672	8	3	10.672	8	3	3.097	9	3	8.267	
3	1	5.375	9	4	11.443	8	4	11.443	8	4	11.443	8	4	11.443	8	4	11.443	8	4	3.511	9	4	9.057	
3	2	6.247	10	0	9.137	9	0	9.137	9	0	9.137	9	0	9.137	9	0	9.137	9	0	4.651	10	0	6.971	
3	3	7.334	10	1	9.607	9	1	9.607	9	1	9.607	9	1	9.607	9	1	9.607	9	1	5.074	10	1	7.345	
3	4	7.918	10	2	10.490	9	2	10.490	9	2	10.490	9	2	10.490	9	2	10.490	9	2	5.628	10	2	8.139	
3	5	8.700	10	3	11.251	9	3	11.251	9	3	11.251	9	3	11.251	9	3	11.251	9	3	6.314	10	3	8.789	
3	6	9.480	11	0	9.737	10	0	9.737	10	0	9.737	10	0	9.737	10	0	9.737	10	0	6.990	11	0	7.505	
3	7	10.246	11	1	10.197	10	1	10.197	10	1	10.197	10	1	10.197	10	1	10.197	10	1	7.659	11	1	7.874	
3	8	11.229	11	2	11.074	10	2	11.074	10	2	11.074	10	2	11.074	10	2	11.074	10	2	8.322	11	2	8.665	
4	0	5.222	11	3	11.826	10	3	11.826	10	3	11.826	10	3	11.826	10	3	11.826	10	3	8.982	11	3	9.310	
4	1	6.028	12	0	10.331	11	0	10.331	11	0	10.331	11	0	10.331	11	0	10.331	11	0	9.457	12	0	8.036	
4	2	6.878	12	1	10.783	11	1	10.783	11	1	10.783	11	1	10.783	11	1	10.783	11	1	4.099	12	1	8.407	
4	3	7.703	12	2	11.654	11	2	11.654	11	2	11.654	11	2	11.654	11	2	11.654	11	2	4.925	12	2	9.188	
4	4	8.518	13	0	10.921	12	0	10.921	12	0	10.921	12	0	10.921	12	0	10.921	12	0	5.788	13	0	8.564	
4	5	9.293	13	1	11.365	12	1	11.365	12	1	11.365	12	1	11.365	12	1	11.365	12	1	6.450	13	1	8.924	
4	6	10.065	14	0	11.506	13	0	11.506	13	0	11.506	13	0	11.506	13	0	11.506	13	0	7.156	14	0	9.089	

A somewhat more complicated case making use of the possibility of combining component data when some sample sizes are equal, as described in Section 1, is illustrated by

Example 2. Consider a five-component series system with test data  $X_1 = 0, X_2 = 4, X_3 = 2, X_4 = 1, X_5 = 3$ , based on corresponding sample sizes  $n_1 = n_2 = n_3 = 300, n_4 = n_5 = 200$ . Here  $c = (3/300 + 2/200) = .02, a_1 = a_2 = a_3 = 1/300(.02) = 1.6, a_4 = a_5 = 1/200(.02) = 1/4$ . For the equivalent  $k^* = 2$  problem we divide the two distinct values of the  $a_i$ 's by their sum  $c_0 = 1/6 + 1/4 = 5/12$ , obtaining  $a_1^* = .40, a_2^* = .60$ . The corresponding numbers of failures are  $X_1^* = X_1 + X_2 + X_3 = 6$  and  $X_2^* = X_4 + X_5 = 4$ . For the reduced problem we consult Table 1 for  $\alpha = .10$  to find the 90 percent confidence bound 7.564. This must be multiplied by  $c_0$  to obtain the bound for the original  $\theta(\lambda)$ . The 90 percent lower bound  $r(x)$  for the system reliability given by (1.6) thus becomes  $1 - cc_0 t(x) = 1 - (.02)(5/12)(7.564) = .937$ .

If the observed numbers of failures fall outside the limits of Table 1, the approximate methods discussed in the next section may be employed.

#### 4. Approximations

In this section approximate confidence bounds for systems with more than two components having distinct sample sizes are developed. The use of the maximum likelihood ratio confidence bound for cases falling outside the scope of Table 1 is also discussed.

For  $k > 2$  it is impractical to generate the optimal ordering and the corresponding values of the upper confidence bound for more than a few illustrative cases. Thus, some method of approximating solutions for  $k > 2$  with acceptable precision is required. The approach to be followed here is to find a  $k = 2$  problem which is sufficiently similar in structure to the given  $k > 2$  problem so that the confidence bounds for the two problems are essentially the same except for a normalizing factor. This method may be thought of as an extension and refinement of the Lindstrom-Madden procedure (see Lloyd and Lipow 1977, and cf. Harris and Soms 1980).

The Lindstrom-Madden method first estimates the reliability by maximum likelihood and then uses the  $k = 1$  confidence bound solution for the component with the smallest sample size and a fictitious number of failures determined so as to reproduce the estimated system reliability. The procedure proposed here is to estimate two quantities, the value of  $\theta$  and the variance of the maximum likelihood estimate of  $\theta$ , and to use these estimates to find a  $k = 2$  problem based on two of the original  $a_i$ 's with a pair of corresponding fictitious observation values chosen to reproduce the estimated quantities. The two  $a_i$ 's are chosen to be as large as possible (corresponding to sample sizes as small as possible) subject to two constraints. The

first constraint guarantees that the resulting fictitious observations are non-negative. The second constraint requires that  $a_i$ 's corresponding to zero failures in the original problem not be considered unless all but one of the components have zero failures. These considerations lead to a unique  $k = 2$  problem whose solution provides a very good approximation to the solution for the original  $k > 2$  problem.

The elimination of  $a_i$ 's for components exhibiting zero failures is justified by the fact that the maximum likelihood ratio confidence bound (discussed later) is invariant under such transformations. That is, if the dimension  $k$  is reduced by the elimination of all  $a_i$ 's corresponding to zero failures, then the value of the maximum likelihood ratio bound remains unchanged.

The choice of the two quantities whose estimates are used to determine the pair of fictitious observations is supported by analogy with Lindstrom-Madden (in the case of  $\theta$ ) and by the fact that the two estimates are the ingredients of the asymptotic maximum likelihood ratio confidence bound thereby insuring asymptotic optimality. The details of the approximation algorithm are as follows:

(a) First  $\theta(\lambda) = \sum_{i=1}^k a_i \lambda_i$  is estimated by

$$\hat{\theta} = \sum_{i=1}^k a_i X_i, \quad (4.1)$$

and the quantity  $\text{Var}(\hat{\theta}) = \sum_{i=1}^k a_i^2 \lambda_i$  is estimated by

$$\hat{v} = \sum_{i=1}^k a_i^2 X_i. \quad (4.2)$$



(b) Next if at least one component exhibits one or more failures, the pair  $(a_i, a_j)$ ,  $i < j$ , is selected so that

$$(i) \quad a_i \leq \hat{v}/\hat{\theta} \quad , \quad \text{and}$$

$$(ii) \quad a_j \geq \hat{v}/\hat{\theta} \quad .$$

Subject to these conditions,  $a_i$  and  $a_j$  are taken to be the largest available values associated with at least one failure. If all  $a_i$ 's satisfying (i) correspond to zero failures, then  $a_i$  is taken to be the largest in that group. If all the  $a_j$ 's satisfying (ii) correspond to zero failures, then  $a_j$  is taken to be  $a_k$ . If all components exhibit zero failures,  $\hat{v}/\hat{\theta}$  is indeterminate and  $a_i$  and  $a_j$  are taken to be  $a_1$  and  $a_k$  respectively.<sup>1</sup>

(c) The pseudo-observations  $x_1^*$  and  $x_2^*$  are computed by the formulae

$$x_1^* = \frac{a_j \hat{\theta} - \hat{v}}{a_i (a_j - a_i)} \quad , \quad (4.3)$$

$$x_2^* = \frac{\hat{v} - a_i \hat{\theta}}{a_j (a_j - a_i)} \quad .$$

These values will be non-negative by conditions (i) and (ii) of (b), and when associated with  $a_i$  and  $a_j$  respectively they

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<sup>1</sup>For this case the resulting confidence bound is exact and is the same as would be obtained by multiplying  $a_k$  times the upper confidence bound for a single Poisson parameter when zero failures are observed, i.e.,  $t(0) = a_k \log(1/\alpha)$ .

reproduce the values of  $\hat{\theta}$  and  $\hat{v}$  provided all other observations are replaced by zeros.

- (d) The  $k = 2$  problem with  $x_1^*$  and  $x_2^*$  associated with  $a_1^* = a_i/(a_i + a_j)$  and  $a_2^* = a_j/(a_i + a_j)$  respectively may now be treated using Table 1 to yield  $t(x_1^*, x_2^*)$ . Since  $x_1^*$  and  $x_2^*$  are not necessarily integers, it may be necessary to interpolate with respect to these arguments as well as the value of  $a_1 = a_1^*$ .

- (e) The approximate upper confidence bound  $t^*$  for  $\theta$  for the original  $k > 2$  problem is then given by

$$t^*(x) = (a_i + a_j)t(x_1^*, x_2^*) \quad . \quad (4.4)$$

In order to check the validity of this approximation algorithm, the confidence bounds for the first 24 points in the optimal (two-stage prospective) ordering were computed for a typical  $k = 3$  case. These results were obtained by a rather laborious method based on formula (2.7) and involving repeated interactive searches of the  $(\lambda_1, \lambda_2, \lambda_3)$  simplex. The approximation algorithm was applied to each of these points and the comparative results are shown in Table 2. The values of  $a_1$ ,  $a_2$ , and  $a_3$  for this example were chosen to be of roughly the same magnitude but not so close that the combination of any two would be indicated.

To check the algorithm for cases farther from the origin several additional examples were considered using a somewhat different method which avoids the necessity for sequentially generating the

Table 2. The Performance of the Approximation Algorithm for the First 24 Ordered Points for a Typical Example With  $k = 3$ .

$\alpha = .05; a_1 = .2, a_2 = .3, a_3 = .5$						
Two-Step Prospective Sequential					Approximation Algorithm	Relative Error
n	$x_1$	$x_2$	$x_3$	$t(x)$	$t^*(x)$	
1	0	0	0	1.498	1.498	+ 00.0 %
2	1	0	0	1.553	1.555	+ 00.1
3	0	1	0	1.656	1.663	+ 00.4
4	2	0	0	1.703	1.705	+ 00.2
5	1	1	0	1.756	1.540	- 12.3
6	2	1	0	1.861	1.752	- 05.9
7	3	0	0	1.933	1.891	- 02.1
8	0	2	0	1.981	1.995	+ 00.7
9	1	2	0	2.085	2.052	- 01.6
10	4	0	0	2.088	2.094	+ 00.3
11	3	1	0	2.162	1.991	- 07.9
12	2	2	0	2.268	2.208	- 02.7
13	5	0	0	2.300	2.309	+ 00.4
14	0	3	0	2.354	2.397	+ 01.8
15	0	0	1	2.372	2.372	+ 00.0
16	4	1	0	2.382	2.291	- 03.9
17	1	0	1	2.429	2.431	+ 00.1
18	1	3	0	2.472	2.438	- 01.4
19	5	1	0	2.504	2.486	- 00.7
20	0	1	1	2.543	2.529	- 00.5
21	6	0	0	2.575	2.551	- 01.0
22	2	0	1	2.602	2.589	- 00.8
23	2	3	0	2.609	2.641	+ 01.2
24	1	1	1	2.660	2.668	+ 00.3

optimal ordering of the sample points. For these examples the ordering was generated by the values of the maximum likelihood ratio confidence bounds (see below) associated with the sample points. Since this ordering is asymptotically optimal (for large  $\lambda_i$ 's), it can be expected to produce good results for sample points well removed from the origin. The values of the confidence bounds calculated using (2.7) and the approximations obtained by the proposed algorithm are shown for these examples in Table 3.

In Table 3 the tendency of the computed values of  $t(x)$  to be slightly larger than the algorithm values may be due to the fact that the ordering used to compute the former is non-optimal. Formal application of the algorithm may occasionally result in the selection of nearly equal values  $a_i$  and  $a_j$  in step (b). When this happens, improved results may be obtained by first reducing the dimension  $k$  by combining the nearly equal  $a_i$ 's and then applying the algorithm. This method was used for the two cases in Table 3 marked by (+). The algorithm values for all cases where  $a = (.14, .16, .70)$  or  $a = (.15, .41, .44)$  do not differ substantially from the values which would be obtained by reducing to the  $k = 2$  case by combining the nearly equal  $a_i$ 's. Similarly, the values for the cases where  $a = (.32, .33, .35)$  can be nearly reproduced by multiplying the  $k = 1$  bound by  $\bar{a} = .333$ .

The number of sample points appearing in the ordering before a given level of  $t(x)$  is reached increases rapidly as  $k$  increases.

Table 3. Several Examples Comparing the Algorithm Values With the Exact Bounds Based on the Ordering Generated by the m&rb.

$\alpha = .05$							
$a_1$	$a_2$	$a_3$	$x_1$ $x_2$ $x_3$	Position in Ordering	$t(x)$	Algorithm Value $t^*(x)$	m&rb
.20, .30, .50			2, 2, 1	50	3.329	3.273	3.070
			5, 0, 2	100	4.068	3.957	3.798
			6, 6, 0	200	4.887	4.789	4.708
			9, 1, 3	400	5.980	5.892	5.795
.14, .16, .70			5, 4, 0	50	3.026	3.084 <sup>†</sup>	2.218
			2, 10, 0	100	3.424	3.531 <sup>†</sup>	2.921
			2, 5, 1	200	4.407	4.086	3.729
.15, .41, .44			13, 0, 0	50	3.245	3.223	2.980
			5, 2, 1	100	4.028	3.860	3.699
			3, 3, 2	200	4.861	4.738	4.559
.32, .33, .35			0, 3, 2	50	3.492	3.518	3.256
			1, 5, 1	100	4.339	4.332	4.076
			1, 5, 3	200	5.198	5.234	4.993

<sup>†</sup> Reduction to  $k=2$  case by combining  $a_1$  and  $a_2$ .

Hence for  $k > 2$  the algorithm may be applied to sample points positioned quite far along in the ordering without requiring values beyond the scope of Table 1, as is seen in the examples of Table 3.

The results detailed in Tables 2 and 3 suggest that the application of the proposed algorithm together with the combining of nearly equal  $a_i$ 's when indicated will nearly always produce confidence bounds for  $\theta(\lambda)$  subject to relative errors not exceeding 10 percent and often much less. Furthermore, the lower confidence bound for reliability  $r(x)$  given by (1.6) will exhibit a much smaller relative error since the relative error in approximations for  $t(x)$  applies only to the difference between the lower bound and one; a quantity which at worst is of the order of  $1/10$  in the contemplated applications.

In situations where very large sample sizes are available, it may happen that the observed numbers of failures exceed the limits of Table 1 even though the system reliability is high. For such cases the maximum likelihood ratio bound (mlrb) may be used. This approximate confidence bound is obtained in the usual way from the maximum likelihood ratio statistic for testing the hypothesis  $H_0 : \theta(\lambda) = \theta_0$  versus all alternatives. The corresponding one-sided confidence bound may be shown (see Johns 1975) to be determined as follows: Let  $\hat{\mu}$  be the positive real root less than  $1/(\text{largest } a_i \text{ for which } X_i > 0)$  of the equation

$$\chi_{1,2\alpha}^2 = 2 \sum_{i=1}^k X_i \left\{ \frac{a_i \hat{\mu}}{1 - a_i \hat{\mu}} + \log(1 - a_i \hat{\mu}) \right\}, \quad (4.5)$$

where  $\chi_{1,2\alpha}^2$  is the upper  $100(2\alpha)$ -th percentage point of the chi-squared distribution with one degree of freedom. Then the quantity

$$\hat{t}(X) = \sum_{i=1}^k a_i X_i / (1 - a_i \hat{\mu}) \quad (4.6)$$

is the approximate upper  $1 - \alpha$  level confidence bound for  $\theta(\underline{\lambda})$ . As  $\max(\lambda_1, \lambda_2, \dots, \lambda_n) \rightarrow \infty$ , the mlrb  $\hat{t}(X)$  may be shown (see Johns 1975) to be asymptotically equivalent to

$$\tilde{t}(X) = \sum_{i=1}^k a_i X_i + z_{\alpha} \left( \sum_{i=1}^k a_i^2 X_i \right)^{1/2}, \quad (4.7)$$

where  $z_{\alpha}$  is the  $100\alpha$ -th percentage point of the standard normal distribution.

Neither of these approximate bounds is useful for sample points near the origin in the usual orderings, but  $\hat{t}$  given by (4.6) becomes sufficiently precise for application to sample points beyond the scope of Table 1. A comparison of  $t(x_1, x_2)$  and the corresponding mlrb for the last (i.e., the 100-th) points in each of the optimal orderings for the cases covered in Table 1 is given in Table 4.

Table 4.  $m\text{lr}b/t(x_1, x_2)$  for the 100-th Point in Each Ordering

$\alpha$	$a_1$								
	.05	.10	.15	.20	.25	.30	.35	.40	.45
.01	.62	.95	.97	.97	.98	.97	.98	.97	.99
.05	.72	.88	.96	.95	.98	.97	.99	.95	.96
.10	.89	.95	.95	.96	.97	1.00	.97	.96	.98

These results suggest that the  $m\text{lr}b$  possesses satisfactory precision for sample points beyond those listed in Table 1 for  $k = 2$  whenever  $a_1 \geq .10$ . The last column of Table 3 giving the  $m\text{lr}b$  values for the examples considered illustrates the fact that for  $k > 2$  the  $m\text{lr}b$  tends to underestimate the correct value of the bound for sample points within the range that can be dealt with using Table 1 and the algorithm.

Two potential sources of error for the lower confidence bound on system reliability remain to be discussed. They are (i) the Poisson approximation to the binomial distribution of the observed failures, and (ii) the approximation for reliability given in (1.2) and reflected in the formula (1.6) for the lower bound  $r(x)$ . It is intuitively clear from (1.1) ff. that the "worst case" for the Poisson approximation should occur when  $k = 1$ ; since to match a given  $k = 1$  level of reliability, say  $1 - p$ , by a  $k > 1$  case, we must have  $p \approx \sum_{i=1}^k p_i$ , so that the  $p_i$ 's must be smaller than  $p$  which tends to improve the Poisson approximation.



For the case  $k = 1$  the familiar upper confidence bounds for a single Poisson parameter apply and the actual coverage probabilities for the proposed method (3.1) can be computed for any  $n$  and  $p$  from tables of the binomial distribution. The results of several such calculations are shown in Table 5.

Table 5. "Worst Case" Analysis ( $k=1$ ). Minimum Coverage Probabilities for  $r(x)$ .

Reliability $q = 1 - p$	$1 - \alpha$		
	.90	.95	.99
.95	.906	.954	.991
.90	.912	.958	.992
.80	.924	.965	.995
.70	.936	.972	.997

These values suggest that the approximations operate to make the proposed confidence bounds slightly conservative. It is interesting to observe that the minimum coverage probabilities are not drastically different from the nominal values, even for a true reliability as low as .70.

##### 5. Acknowledgments and Remarks Concerning the Computations

The computations for Table 1 were performed on a Digital Equipment PDP-11/34 running under a UNIX operating system. The production program for these computations (506 lines) was written in the "C" language by the author. The method used for the

computation of the upper bounds was an implementation of (2.7); and as has been noted, the sample points were ordered by the two-step prospective sequential method. The tabled results were subjected to various checks to insure that the correct global maxima were found in each case.

The computations involved in obtaining the one-step look-ahead results and the tree analysis associated with the semi-Bayesian results discussed in Section 3 were done on an IBM 370/168 machine using Fortran programs developed by David Pasta. These programs compute the upper bounds by a somewhat more complicated method used in the earlier phases of the study and detailed in Johns (1975).

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generated sequentially by a two-stage, prospective optimization procedure. A generalization of the Lindstrom-Madden technique is proposed for using the tables to find confidence bounds for systems consisting of more than two components with differing sample sizes.